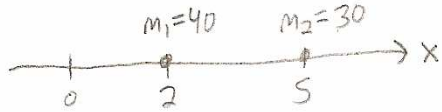
8.3 homework #21–33 odd

21. 

Find the moment M of the system about the origin and the center of mass $\bar{x}$.

The moment M (M_y) of the system about the origin: $M = \sum_{i=1}^{2} m_i x_i = m_1x_1 + m_2x_2$

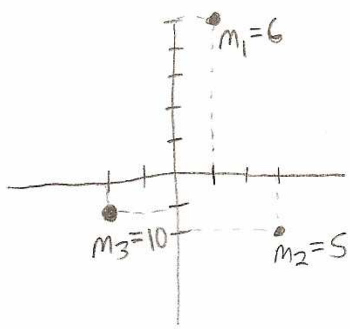
$= 40\cdot 2 + 30\cdot 5 = \boxed{230.}$

The center of mass $\bar{x} = \dfrac{\sum m_i x_i}{m} = \dfrac{M}{m} = \dfrac{230}{70} = \boxed{3\tfrac{2}{7}.}$

total mass

23. $m_1=6$, $m_2=5$, $m_3=10$
$P_1(1,5)$ $P_2(3,-2)$ $P_3=(-2,-1)$

Find the moments M_x and M_y and the center of mass of the system.



M_x = moment of the system about the x axis (measure of the tendency of the system to rotate about the x axis)

$= \sum_{i=1}^{3} m_i y_i = m_1y_1 + m_2y_2 + m_3y_3$

$= 6\cdot 5 + 5\cdot(-2) + 10\cdot(-1) = 30 - 10 - 10 = \boxed{10 = M_x}$

M_y = moment of the system about the y axis $= \sum_{i=1}^{3} m_i x_i = m_1x_1 + m_2x_2 + m_3x_3$

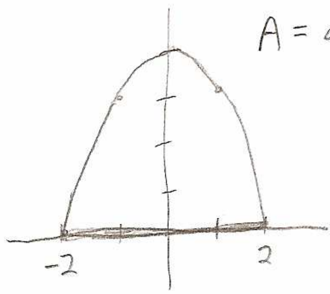
$= 6\cdot 1 + 5\cdot 3 + 10\cdot(-2) = 6 + 15 - 20 = \boxed{1 = M_y}$

(measure of the tendency of the system to rotate about the y-axis)

Center of Mass $= (\bar{x}, \bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \boxed{\left(\dfrac{1}{21}, \dfrac{10}{21}\right)}.$

total mass

25. Find the centroid (center of mass) of the region bounded by $y = 4 - x^2$ and $y = 0$.



$$A = \text{area} = \int_{-2}^{2} (4-x^2)\,dx = 2\int_0^2 (4-x^2)\,dx = 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2\left[8 - \frac{8}{3}\right]$$

$$A = 2\left[\frac{24-8}{3}\right] = 2\left[\frac{16}{3}\right] = \frac{32}{3}.$$

$$\bar{x} = \frac{1}{A}\int_{-2}^{2} x\,f(x)\,dx = 0$$ since the region is centered on the y axis. (No need to do the integral.)

$$\bar{y} = \frac{1}{A}\int_{-2}^{2} \frac{1}{2}[f(x)]^2\,dx = \frac{1}{\left(\frac{32}{3}\right)} \cdot \cancel{2}\int_0^2 \frac{1}{\cancel{2}}\left[4-x^2\right]^2 dx = \frac{3}{32}\int_0^2 \left[16 - 8x^2 + x^4\right] dx$$

$$= \frac{3}{32}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2 = \frac{3}{32}\left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{3}{32}\left[\frac{480 - 320 + 96}{15}\right]$$

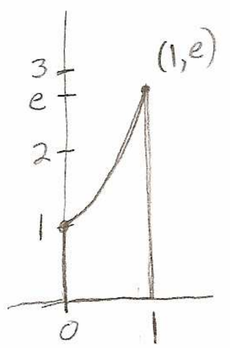
32
×15
160
320
480

$$= \frac{\cancel{3}}{32}\left[\frac{\overset{8}{\cancel{256}}}{\underset{5}{\cancel{15}}}\right] = \frac{8}{5}.$$

32
×8
256

So, the centroid is $\boxed{\left(0,\ 1\tfrac{3}{5}\right)}$.

27. Find the centroid of the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.



$$\text{Area} = A = \int_0^1 e^x\,dx = e^x\Big]_0^1 = e - 1.$$

$$\bar{x} = \frac{1}{A}\int_0^1 x\,f(x)\,dx = \frac{1}{e-1}\int_0^1 x e^x\,dx = \frac{1}{e-1}\left[xe^x - \int e^x\,dx\right]$$

$u = x \qquad v = e^x$

$du = dx \quad dv = e^x\,dx$

$$= \frac{1}{e-1}\left[xe^x - e^x\right]_0^1 = \frac{1}{e-1}\left[\overset{0}{(\cancel{e} - \cancel{e})} - (0-1)\right] = \frac{1}{e-1}.$$

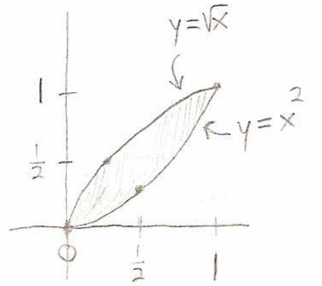
$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}[f(x)]^2\,dx = \frac{1}{e-1}\cdot\frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{2(e-1)}\left[\frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{1}{2(e-1)}\left[\frac{e^2}{2} - \frac{1}{2}\right] = \frac{1}{2(e-1)}\cdot\frac{e^2-1}{2} = \frac{1}{2\cancel{(e-1)}}\cdot\frac{(e+1)\cancel{(e-1)}}{2} = \frac{e+1}{4}.$$

So, the centroid is $\boxed{\left(\frac{1}{e-1},\ \frac{e+1}{4}\right)}$.

29. Find the centroid of the region bounded by the given curves.

$y=x^2,\ x=y^2$ $\quad A=\int_0^1\left(x^{\frac{1}{2}}-x^2\right)dx=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{3}\right]_0^1=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}.$



$$\bar{x}=\frac{1}{A}\int_0^1 x\left[f(x)-g(x)\right]dx=3\int_0^1 x\left[x^{\frac{1}{2}}-x^2\right]dx$$

$$=3\int_0^1\left(x^{\frac{3}{2}}-x^3\right)dx=3\left[\frac{2}{5}x^{\frac{5}{2}}-\frac{x^4}{4}\right]_0^1=3\left[\frac{2}{5}-\frac{1}{4}\right]=3\left[\frac{8-5}{20}\right]$$

$$=\frac{9}{20}.$$

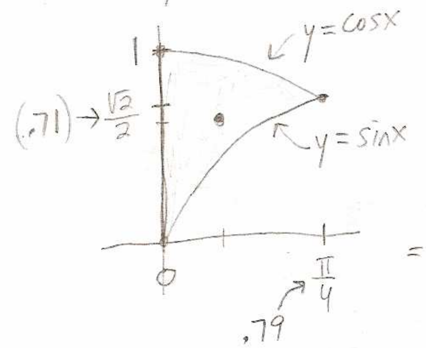
$$\bar{y}=\frac{1}{A}\int_0^1\frac{1}{2}\left\{\left[f(x)\right]^2-\left[g(x)\right]^2\right\}dx=3\int_0^1\frac{1}{2}\left\{\left[\sqrt{x}\right]^2-\left[x^2\right]^2\right\}dx$$

$$=\frac{3}{2}\int_0^1\left[x-x^4\right]dx=\frac{3}{2}\left[\frac{x^2}{2}-\frac{x^5}{5}\right]_0^1=\frac{3}{2}\left[\frac{1}{2}-\frac{1}{5}\right]=\frac{3}{2}\left[\frac{5-2}{10}\right]=\frac{9}{20}.$$

centroid $=\boxed{\left(\frac{9}{20},\frac{9}{20}\right)}$

31. Find the centroid of the region bounded by the given curves.

$y=\sin x,\ y=\cos x,\ x=0,\ x=\frac{\pi}{4}.$ $\quad A=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)\,dx=\left[\sin x+\cos x\right]_0^{\frac{\pi}{4}}$



$$=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)-(0+1)=\frac{2\sqrt{2}}{\cancel{2}}-1=\sqrt{2}-1.$$

$$\bar{x}=\frac{1}{A}\int_0^{\frac{\pi}{4}}x\left[f(x)-g(x)\right]dx=\frac{1}{\sqrt{2}-1}\int_0^{\frac{\pi}{4}}x\left[\cos x-\sin x\right]dx$$

$$=\frac{1}{\sqrt{2}-1}\left[\int_0^{\frac{\pi}{4}}x\cos x\,dx-\int_0^{\frac{\pi}{4}}x\sin x\,dx\right]$$

$\boxed{u=x \quad v=\sin x \quad du=dx \quad dv=\cos x\,dx}$ $\quad \boxed{u=x \quad v=-\cos x \quad du=dx \quad dv=\sin x\,dx}$

$$=\frac{1}{\sqrt{2}-1}\left[\left(x\sin x-\int\sin x\,dx\right)-\left(-x\cos x+\int\cos x\,dx\right)\right]=\frac{1}{\sqrt{2}-1}\left[x\sin x+\cos x+x\cos x-\sin x\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{\sqrt{2}-1}\left[\left(\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}+\cancel{\frac{\sqrt{2}}{2}}+\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}-\cancel{\frac{\sqrt{2}}{2}}\right)-(0+1+0-0)\right]=\frac{1}{\sqrt{2}-1}\left[\frac{\sqrt{2}\pi}{8}+\frac{\sqrt{2}\pi}{8}-1\right]$$

$$=\frac{1}{\sqrt{2}-1}\left[\frac{\cancel{2}\sqrt{2}\pi}{\cancel{8}_{4}}-1\right]=\frac{1}{\sqrt{2}-1}\left[\frac{\sqrt{2}\pi}{4}-1\right]\approx .27$$

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31. continued

$$\bar{y} = \frac{1}{A}\int_0^{\frac{\pi}{4}} \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx = \frac{1}{\sqrt{2}-1}\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}}(\cos^2 x - \sin^2 x)\,dx$$

$$\frac{1}{2(\sqrt{2}-1)}\int_0^{\frac{\pi}{4}}\cos 2x\,dx = \frac{1}{2(\sqrt{2}-1)}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{4(\sqrt{2}-1)}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= \frac{1}{4(\sqrt{2}-1)}[1-0] = \frac{1}{4(\sqrt{2}-1)} \approx .60$$

$$\text{centroid} = \boxed{\left(\frac{1}{\sqrt{2}-1}\cdot\underbrace{\left[\frac{\sqrt{2}\pi}{4}-1\right]}_{\frac{\sqrt{2}\pi-4}{4}},\ \frac{1}{4(\sqrt{2}-1)}\right)} = \boxed{\left(\frac{\sqrt{2}\pi-4}{4(\sqrt{2}-1)},\ \frac{1}{4(\sqrt{2}-1)}\right)}$$

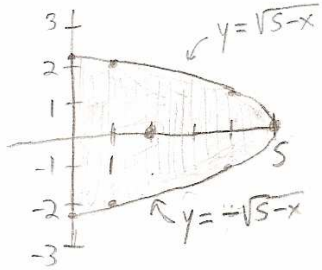
33. Find the centroid of the region bounded by $x = 5-y^2,\ x=0.$

$y^2 = 5-x \Rightarrow y = \pm\sqrt{5-x}$

x	y
0	$\pm\sqrt{5}$
1	± 2
5	0
4	± 1

$$A = \int_{-\sqrt{5}}^{\sqrt{5}}(5-y^2)\,dy = 2\int_0^{\sqrt{5}}(5-y^2)\,dy$$

$$= 2\left[5y - \frac{y^3}{3}\right]_0^{\sqrt{5}} = 2\left[5\sqrt{5} - \frac{5\sqrt{5}}{3}\right] = 2\cdot\frac{10\sqrt{5}}{3} = \frac{20\sqrt{5}}{3}.$$



$$\bar{x} = \frac{1}{A}\int_0^5 x\left[f(x)-g(x)\right]dx$$

$$\bar{x} = \frac{3}{20\sqrt{5}}\int_0^5 x\left[\sqrt{5-x} - (-\sqrt{5-x})\right]dx = \frac{3}{\cancel{20}_{10}\sqrt{5}}\int_0^5 x\cdot\cancel{2}\sqrt{5-x}\,dx$$

$$= \frac{3}{10\sqrt{5}}\int_0^5 x\sqrt{5-x}\,dx = \frac{-3}{10\sqrt{5}}\int_5^0 (5-u)\,u^{\frac{1}{2}}\,du = \frac{3}{10\sqrt{5}}\int_0^5\left(5u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$u = 5-x \rightarrow x = 5-u$

$du = -dx$

$$= \frac{3}{10\sqrt{5}}\left[5\cdot\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^5$$

$$= \frac{3}{10\sqrt{5}}\left[\frac{10}{3}\cdot 5\sqrt{5} - \frac{2}{5}\cdot 25\sqrt{5}\right] = \frac{3}{10\sqrt{5}}\left[\frac{50\sqrt{5}}{3} - \frac{50\sqrt{5}}{5}\right] = \frac{3}{10\sqrt{5}}\left[\frac{250\sqrt{5}-150\sqrt{5}}{15}\right]$$

$$= \frac{\cancel{3}}{\cancel{10\sqrt{5}}}\left[\frac{\cancel{100\sqrt{5}}^{10}}{\cancel{15}_{5}}\right] = 2.$$

$\bar{y} = 0$ since the region is centered on the x axis.

So, centroid $= \boxed{(2,0)}$