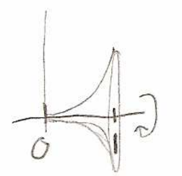
1. $y=x^4,\ 0\le x\le 1$; about the x-axis ~ Part a:

$y'=4x^3$

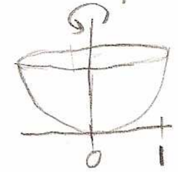
$(y')^2=16x^6$

$$S.A.=\int 2\pi y\,ds=\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\boxed{2\pi\int_0^1 x^4\sqrt{1+16x^6}\,dx}$$



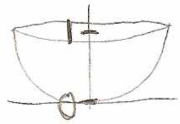
about the y-axis ~ Part b:

$$S.A.=\int 2\pi x\,ds=\int_0^1 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



$$=\boxed{2\pi\int_0^1 x\sqrt{1+16x^6}\,dx}$$

OR (Part b): $$S.A.=\int 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\boxed{2\pi\int_0^1 y^{\frac{1}{4}}\sqrt{1+\frac{1}{16y^{\frac{3}{2}}}}\,dy}$$

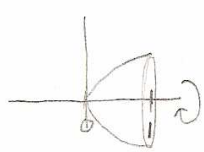


$y=x^4 \Rightarrow x=y^{\frac{1}{4}}$

$\frac{dx}{dy}=\frac{1}{4}y^{-\frac{3}{4}}=\frac{1}{4y^{\frac{3}{4}}}.\quad \left(\frac{dx}{dy}\right)^2=\frac{1}{16y^{\frac{3}{2}}}$

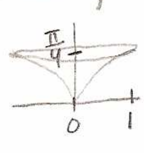
3. $y=\tan^{-1}x,\ 0\le x\le 1$

a) about x-axis: $y'=\frac{1}{1+x^2}\quad (y')^2=\frac{1}{(1+x^2)^2}.\quad S.A.=\int 2\pi y\,ds=\int 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$



$$=\boxed{2\pi\int_0^1 \tan^{-1}x\sqrt{1+\frac{1}{(1+x^2)^2}}\,dx}$$

b) about y-axis: $$S.A.=\int 2\pi x\,ds=\int 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\boxed{2\pi\int_0^1 x\sqrt{1+\frac{1}{(1+x^2)^2}}\,dx}$$



$\boxed{\text{OR}}$ $$S.A.=\int 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\boxed{2\pi\int_0^{\frac{\pi}{4}}\tan y\sqrt{1+\sec^4 y}\,dy}$$

$y=\tan^{-1}x$

$x=\tan y$

$\frac{dx}{dy}=\sec^2 y$

$\left(\frac{dx}{dy}\right)^2=\sec^4 y$

5. $y=x^3$, $0\le x\le 2$, about x axis. Find Surface Area. $u=1+9x^4$, $du=36x^3$.  $<16$ $\frac{9}{144}$

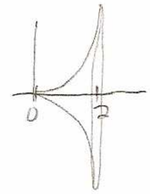
$y'=3x^2$ S.A. $=\int_0^2 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \frac{1}{36}2\pi\int_0^2 x^3\sqrt{1+9x^4}\,dx\,36 = \frac{\pi}{18}\int_1^{145}u^{\frac{1}{2}}\,du$

| x | u |
|---|---|
| 0 | 1 |
| 2 | 145 |

$(y')^2=9x^4$.

$=\frac{\pi}{\cancel{18}_9}\frac{\cancel{2}}{3}\left[u^{\frac{3}{2}}\right]_1^{145} = \frac{\pi}{27}\left[145^{\frac{3}{2}}-1\right] = \boxed{\frac{\pi}{27}\left[145\sqrt{145}-1\right]}$

145
$5\cdot 29$




7. $y=\sqrt{1+4x}$, $1\le x\le 5$, about x axis. Find S.A.

$y'=\frac{1}{2}(1+4x)^{-\frac{1}{2}}\cdot 4=\frac{2}{\sqrt{1+4x}}$. S.A. $=\int 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^5\sqrt{1+4x}\sqrt{1+\frac{4}{1+4x}}\,dx$

$(y')^2=\frac{4}{1+4x}$. $=2\pi\int_1^5\sqrt{1+4x}\sqrt{\frac{1+4x+4}{1+4x}}\,dx = 2\pi\int_1^5 \cancel{\sqrt{1+4x}}\cdot\frac{\sqrt{5+4x}}{\cancel{\sqrt{1+4x}}}\,dx$

$=\frac{1}{4}\cdot 2\pi\int_1^5\sqrt{5+4x}\,dx\,(4) = \frac{\pi}{2}\int_9^{25}u^{\frac{1}{2}}\,du = \frac{\pi}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\left[u^{\frac{3}{2}}\right]_9^{25} = \frac{\pi}{3}\left[25^{\frac{3}{2}}-9^{\frac{3}{2}}\right]=\frac{\pi}{3}[125-27]$

$=\boxed{\frac{98\pi}{3}}$

$u=5+4x$, $du=4\,dx$

9. $y=\sin\pi x$, $0\le x\le 1$, about x axis

$y'=\pi\cos\pi x$ S.A. $=\int 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \frac{-1}{\pi^2}\cdot 2\pi\int_0^1\sin\pi x\sqrt{1+\pi^2\cos^2\pi x}\,dx\,(-\pi^2)$

$(y')^2=\pi^2\cos^2\pi x$  $u=\pi\cos\pi x$, $du=-\pi^2\sin\pi x\,dx$

| x | u |
|---|---|
| 0 | $\pi$ |
| 1 | $-\pi$ |

$=\frac{-2}{\pi}\int_{\pi}^{-\pi}\sqrt{1+u^2}\,du = \frac{2}{\pi}\int_{-\pi}^{\pi}\sqrt{1+u^2}\,du \overset{\text{even function}}{=} \frac{2}{\pi}\cdot 2\int_0^{\pi}\sqrt{1+u^2}\,du$

$u=\tan\theta$
$du=\sec^2\theta\,d\theta$

$=\frac{4}{\pi}\int\sec^3\theta\,d\theta = \frac{\overset{2}{\cancel{4}}}{\pi}\cdot\frac{1}{\cancel{2}}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)$

$=\frac{2}{\pi}\left[u\sqrt{1+u^2}+\ln\left|\sqrt{1+u^2}+u\right|\right]_0^{\pi}$

$=\frac{2}{\pi}\left[\pi\sqrt{1+\pi^2}+\ln\left|\sqrt{1+\pi^2}+\pi\right|-(0+0)\right] = \boxed{2\sqrt{1+\pi^2}+\frac{2}{\pi}\ln\left|\sqrt{1+\pi^2}+\pi\right|}$

8.2 homework continued



11. $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}$, $1 \le y \le 2$, about x-axis. ← $SA = \int 2\pi y \, ds$

$\frac{dx}{dy} = \frac{1}{3} \cdot \frac{3}{2}(y^2+2)^{\frac{1}{2}} \cdot 2y = y\sqrt{y^2+2}$. $\left(\frac{dx}{dy}\right)^2 = y^2(y^2+2) = y^4 + 2y^2$.

S.A. $= \int 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_1^2 y\sqrt{1+y^4+2y^2}\, dy = 2\pi \int_1^2 y\sqrt{(y^2+1)^2}\, dy$

$= 2\pi \int_1^2 y(y^2+1)\, dy = 2\pi \int_1^2 (y^3+y)\, dy = 2\pi \left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_1^2 = 2\pi\left[(4+2) - \left(\frac{1}{4}+\frac{1}{2}\right)\right]$

$= 2\pi\left[6 - \frac{3}{4}\right] = 2\pi \cdot \frac{21}{4} = \boxed{\frac{21\pi}{2}}$

13. $y = \sqrt[3]{x}$, $1 \le y \le 2$, about y axis.

$x = y^3$  S.A. $= \int 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \cdot \frac{1}{36}\int_1^2 y^3 \sqrt{1+9y^4}\, dy \cdot 36 = \frac{\pi}{18}\int_{10}^{145} u^{\frac{1}{2}}\, du$

$u = 1+9y^4$
$du = 36y^3\, dy$

| y | u |
|---|---|
| 1 | 10 |
| 2 | 145 |

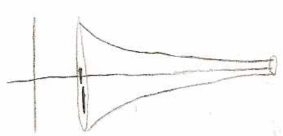
$\frac{dx}{dy} = 3y^2$

$\left(\frac{dx}{dy}\right)^2 = 9y^4$.

$= \frac{\pi}{18} \cdot \frac{2}{3}\left[u^{\frac{3}{2}}\right]_{10}^{145} = \frac{\pi}{27}\left[145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right] = \boxed{\frac{\pi}{27}\left[145\sqrt{145} - 10\sqrt{10}\right]}$

25. $R = \{(x,y) \mid x \ge 1,\ 0 \le y \le \frac{1}{x}\}$ about the x axis. Show S.A. is infinite.

$y = \frac{1}{x}$, $y' = -\frac{1}{x^2}$, $(y')^2 = \frac{1}{x^4}$. S.A. $= \int 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$



$= 2\pi \int_1^\infty \frac{1}{x} \cdot \sqrt{1+\frac{1}{x^4}}\, dx = 2\pi \lim_{t\to\infty} \int_1^t \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\, dx$

$= 2\pi \lim_{t\to\infty} \int_1^t \frac{1}{x} \cdot \frac{\sqrt{x^4+1}}{x^2}\, dx = 2\pi \lim_{t\to\infty} \int_1^t \frac{\sqrt{x^4+1}}{x^3}\, dx$   $x^2 = \tan\theta$, $2x\, dx = \sec^2\theta\, d\theta$

$= \frac{1}{2} \cdot 2\pi \lim_{t\to\infty} \int \frac{\sqrt{x^4+1}}{x^4} \cdot 2x\, dx = \pi \lim_{t\to\infty} \int \frac{\sec\theta}{\tan^2\theta} \cdot \sec^2\theta\, d\theta = \pi \lim_{t\to\infty} \int \frac{\sec\theta\,(1+\tan^2\theta)}{\tan^2\theta}\, d\theta$

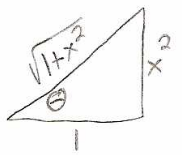
$= \pi \lim_{t\to\infty} \int \left[\frac{\sec\theta}{\tan^2\theta} + \sec\theta\right] d\theta = \pi \lim_{t\to\infty} \int \left[\frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} + \sec\theta\right] d\theta$

$= \pi \lim_{t\to\infty} \left[\int \frac{\cos\theta}{\sin^2\theta}\, d\theta + \int \sec\theta\, d\theta\right] = \pi \lim_{t\to\infty} \left[\int u^{-2}\, du + \int \sec\theta\, d\theta\right]$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$= \pi \lim_{t\to\infty} \left[\frac{u^{-1}}{-1} + \ln|\sec\theta + \tan\theta|\right] =$ (next page)

$$= \pi \lim_{t\to\infty}\left[\frac{-1}{\sin\theta} + \ln|\sec\theta + \tan\theta|\right] = \pi \lim_{t\to\infty}\left[-\frac{\sqrt{1+x^2}}{x^2} + \ln\left|\sqrt{1+x^2} + x^2\right|\right]_1^t$$




$$= \pi \lim_{t\to\infty}\left[-\frac{\sqrt{1+t^2}}{t^2} + \ln\left|\sqrt{1+t^2}+t^2\right| - \left(-\sqrt{2} + \ln\left|\sqrt{2}+1\right|\right)\right]$$

$$= \pi\left[-1 + \infty + \sqrt{2} - \ln\left|\sqrt{2}+1\right|\right] = \infty.$$ Divergent. ∴ S.A. is infinite.

$\boxed{\text{OR}}$

An alternate approach to evaluating $2\pi \lim_{t\to\infty}\int_1^t \frac{\sqrt{x^4+1}}{x^3}\,dx$ : (Using the Comparison Theorem from section 7.8)

$\frac{\sqrt{x^4+1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{x^2}{x^3} = \frac{1}{x}$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, by the Comparison Theorem in 7.8

we have that $\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$ also diverges. ∴ S.A. is infinite.

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Formulas for the Area of a Surface of Revolution:

- For revolution about the x-axis: $S.A. = \int 2\pi y\,ds$

- For revolution about the y-axis: $S.A. = \int 2\pi x\,ds$

Where $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ OR $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ (either ds works)