8.1 homework

Arc Length: $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

or $L = \int_c^d \sqrt{1+[g'(y)]^2}\,dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

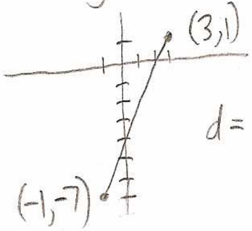
1. $y = 2x-5,\ -1 \le x \le 3.$

$y' = 2,\ (y')^2 = 4.\quad L = \int_{-1}^{3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-1}^{3} \sqrt{1+4}\,dx = \sqrt{5}\,x\Big]_{-1}^{3} = \sqrt{5}\,[3-(-1)]$

$= \boxed{4\sqrt{5}.}$

Check using distance formula:

x	y
-1	-7
3	1



$d = \sqrt{[3-(-1)]^2+[1-(-7)]^2} = \sqrt{16+64} = \sqrt{80} = \sqrt{16\cdot 5} = 4\sqrt{5}.$ ✓

#3 and #5 are on last page

7. $y = 1+6x^{\frac{3}{2}},\ 0 \le x \le 1$

$y' = \overset{3}{\cancel{6}}\cdot\frac{3}{\cancel{2}}x^{\frac{1}{2}} = 9\sqrt{x}.\quad (y')^2 = 81x.\quad L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \frac{1}{81}\int_0^1 \sqrt{1+81x}\,dx^{(81)} = \frac{1}{81}\int_1^{82} u^{\frac{1}{2}}\,du = \frac{1}{81}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{82} = \boxed{\frac{2}{243}\left[82^{\frac{3}{2}}-1\right]}$

$u = 1+81x$

$du = 81\,dx$

x	u
0	1
1	82

OR $\boxed{\frac{2}{243}\left[82\sqrt{82}-1\right]}$

9. $y = \frac{x^5}{6}+\frac{1}{10}x^{-3},\quad 1 \le x \le 2$

$y' = \frac{5x^4}{6}-\frac{3}{10}x^{-4} = \frac{5x^4}{6}-\frac{3}{10x^4}.\quad (y')^2 = \frac{25x^8}{36} - \cancel{2}\cdot\frac{\cancel{5x^4}}{\cancel{6}}\cdot\frac{\cancel{3}}{10\cancel{x^4}} + \frac{9}{100x^8}$

$(y')^2 = \frac{25x^8}{36}-\frac{1}{2}+\frac{9}{100x^8}$

$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1+\frac{25x^8}{36}-\frac{1}{2}+\frac{9}{100x^8}}\,dx = \int_1^2 \sqrt{\frac{25x^8}{36}+\frac{1}{2}+\frac{9}{100x^8}}\,dx$

☆ $= \int_1^2 \sqrt{\left(\frac{5x^4}{6}+\frac{3}{10x^4}\right)^2}\,dx = \int_1^2 \left[\frac{5}{6}x^4+\frac{3}{10}x^{-4}\right]dx = \left[\frac{x^5}{6}-\frac{x^{-3}}{10}\right]_1^2 = \left(\frac{32}{6}-\frac{1}{80}\right)-\left(\frac{1}{6}-\frac{1}{10}\right)$

see next page

$\updownarrow$ $+\frac{8}{80}$

$31 \times 40 = 1240$

$= \frac{31}{6}+\frac{7}{80} = \frac{1240+21}{240} = \boxed{\frac{1261}{240}}$

↗ Notice that:

(From previous problem #9)

$$\left(\frac{5x^4}{6}+\frac{3}{10x^4}\right)\left(\frac{5x^4}{6}+\frac{3}{10x^4}\right)=\frac{25x^8}{36}+\cancel{2}\cdot\frac{\cancel{5x^4}}{\cancel{6}}\cdot\frac{\cancel{3}}{10\cancel{x^4}}+\frac{9}{100x^4}$$

$$=\frac{25x^8}{36}+\frac{1}{2}+\frac{9}{100x^4}.$$

11. $x=\frac{1}{3}\sqrt{y}\,(y-3),\quad 1\le y\le 9.$

$$x=\frac{1}{3}y^{\frac{3}{2}}-y^{\frac{1}{2}}.\qquad \frac{dx}{dy}=\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}y^{\frac{1}{2}}-\frac{1}{2}y^{-\frac{1}{2}}=\frac{1}{2}\sqrt{y}-\frac{1}{2\sqrt{y}}.$$

$$\left(\frac{dx}{dy}\right)^2=\frac{1}{4}y-\cancel{2}\cdot\frac{1}{\cancel{2}}\cancel{\sqrt{y}}\cdot\frac{1}{2\cancel{\sqrt{y}}}+\frac{1}{4y}=\frac{1}{4}y-\frac{1}{2}+\frac{1}{4y}.$$

$$L=\int_1^9\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_1^9\sqrt{1+\frac{1}{4}y-\frac{1}{2}+\frac{1}{4y}}\,dy=\int_1^9\sqrt{\frac{1}{4}y+\frac{1}{2}+\frac{1}{4y}}\,dy$$

$$=\int_1^9\sqrt{\left(\frac{1}{2}\sqrt{y}+\frac{1}{2\sqrt{y}}\right)^2}\,dy\longleftarrow\left(\frac{1}{2}\sqrt{y}+\frac{1}{2\sqrt{y}}\right)^2=\frac{1}{4}y+\cancel{2}\cdot\frac{1}{\cancel{2}}\cancel{\sqrt{y}}\cdot\frac{1}{2\cancel{\sqrt{y}}}+\frac{1}{4y}$$

$$=\frac{1}{4}y+\frac{1}{2}+\frac{1}{4y}$$

$$=\int_1^9\left(\frac{1}{2}y^{\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\right)dy$$

$$=\left[\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}y^{\frac{3}{2}}+\frac{1}{\cancel{2}}\cdot\cancel{2}y^{\frac{1}{2}}\right]_1^9=\left[\frac{1}{3}\cdot27+3\right]-\left[\frac{1}{3}+1\right]=[12]-\left[\frac{4}{3}\right]=\boxed{\frac{32}{3}}$$

13. $y=\ln(\sec x),\quad 0\le x\le\frac{\pi}{4}$

$$y'=\frac{1}{\sec x}\cdot\sec x\tan x=\tan x.\qquad (y')^2=\tan^2x$$

$$L=\int_0^{\frac{\pi}{4}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2x}\,dx=\int_0^{\frac{\pi}{4}}\sec x\,dx$$

$$=\Big[\ln|\sec x+\tan x|\Big]_0^{\frac{\pi}{4}}=\ln\left|\sec\frac{\pi}{4}+1\right|-\ln|\sec 0+0|=\ln|\sqrt{2}+1|-\cancelto{0}{\ln 1}$$

$$=\boxed{\ln(\sqrt{2}+1)}$$

8.1 homework continued

15. $y=\ln(1-x^2),\quad 0\le x\le \frac{1}{2}$

$$y'=\frac{1}{1-x^2}\cdot(-2x)=\frac{-2x}{1-x^2}.\quad (y')^2=\frac{4x^2}{(1-x^2)^2}.$$

$$L=\int_0^{\frac12}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{\frac12}\sqrt{1+\frac{4x^2}{(1-x^2)^2}}\,dx=\int_0^{\frac12}\sqrt{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}}\,dx$$

$$=\int_0^{\frac12}\frac{\sqrt{1-2x^2+x^4+4x^2}}{1-x^2}\,dx=\int_0^{\frac12}\frac{\sqrt{1+2x^2+x^4}}{1-x^2}\,dx=\int_0^{\frac12}\frac{\sqrt{(1+x^2)^2}}{1-x^2}\,dx$$

$$=\int_0^{\frac12}\frac{1+x^2}{1-x^2}\,dx=\int_0^{\frac12}\frac{x^2+1}{-x^2+1}\,dx=\int_0^{\frac12}\left[-1+\frac{2}{1-x^2}\right]dx=\Big[-x\Big]_0^{\frac12}+\int_0^{\frac12}\frac{2}{(1+x)(1-x)}\,dx$$

$$\begin{array}{r} -1+\frac{2}{1-x^2} \\ -x^2+1\,\overline{\big)\,x^2+1} \\ \underline{-(x^2-1)} \\ 2 \end{array}$$

$$=-\frac12+\int_0^{\frac12}\left[\frac{A}{1+x}+\frac{B}{1-x}\right]dx$$

$2=A(1-x)+B(1+x)$

$x=1:\ 2=2B\ \boxed{B=1}$

$x=-1:\ 2=2A\ \boxed{A=1}$

$u=1-x$

$du=-dx$

$$=-\frac12+\Big[\ln|1+x|-\ln|1-x|\Big]_0^{\frac12}$$

$$=-\frac12+\left[\left(\ln\frac32-\ln\frac12\right)-\left(\cancel{\ln 1}^{\,0}-\cancel{\ln 1}^{\,0}\right)\right]=-\frac12+\ln\left[\frac{\left(\frac{3}{\cancel{2}}\right)}{\left(\frac{1}{\cancel{2}}\right)}\right]=\boxed{-\frac12+\ln 3}$$

using trig sub. →

17. $y=e^x,\quad 0\le x\le 1$

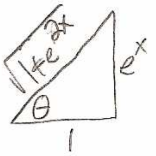
$y'=e^x.\quad (y')^2=e^{2x}.$

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^1\sqrt{1+e^{2x}}\,dx$$

$e^x=\tan\theta,\ e^x dx=\sec^2\theta\, d\theta$

$$dx=\frac{\sec^2\theta\, d\theta}{e^x}$$

$$dx=\frac{\sec^2\theta\, d\theta}{\tan\theta}$$



$$=\int_{\frac{\pi}{4}}^{\tan^{-1}e}\sec\theta\cdot\frac{\sec^2\theta\, d\theta}{\tan\theta}=\int\frac{\sec\theta(1+\tan^2\theta)}{\tan\theta}\,d\theta$$

$$=\int\left[\frac{\sec\theta}{\tan\theta}+\sec\theta\tan\theta\right]d\theta=\int\left[\frac{\frac{1}{\cancel{\cos\theta}}}{\frac{\sin\theta}{\cancel{\cos\theta}}}+\sec\theta\tan\theta\right]d\theta$$

x	θ
0	$\frac{\pi}{4}$
1	$\tan^{-1}e$

$$=\int\left[\csc\theta+\sec\theta\tan\theta\right]d\theta=\Big[\ln|\csc\theta-\cot\theta|+\sec\theta\Big]_{\frac{\pi}{4}}^{\tan^{-1}e}$$

I decided not to use these.

$$=\left[\ln\left|\frac{\sqrt{1+e^{2x}}}{e^x}-\frac{1}{e^x}\right|+\sqrt{1+e^{2x}}\right]_0^1=\ln\left|\frac{\sqrt{1+e^2}-1}{e}\right|+\sqrt{1+e^2}-\left[\ln|\sqrt2-1|+\sqrt2\right]$$

$$=\boxed{\ln|\sqrt{1+e^2}-1|-1+\sqrt{1+e^2}-\ln|\sqrt2-1|-\sqrt2}$$

without trig. substitution (more complex) →

17. $y = e^x, \quad 0 \le x \le 1.$

$y' = e^x, \quad (y')^2 = e^{2x}. \quad L = \int_0^1 \sqrt{1+e^{2x}}\,dx$

$u = e^x$

$du = e^x dx = u\,dx \Rightarrow \frac{du}{u} = dx$

x	u
0	1
1	e

$$= \int_1^e \sqrt{1+u^2}\cdot\frac{du}{u} = \int_1^e \frac{\sqrt{1+u^2}}{u^2}\,\underbrace{du\cdot u}$$

$y = \sqrt{1+u^2}$

$y^2 = 1+u^2 \to \cancel{2}y\,dy = \cancel{2}u\,du$

$y^2 - 1 = u^2$

u	y
1	$\sqrt2$
e	$\sqrt{1+e^2}$

$$= \int_{\sqrt2}^{\sqrt{1+e^2}} \frac{y}{y^2-1}\cdot y\,dy = \int_{\sqrt2}^{\sqrt{1+e^2}} \frac{y^2}{y^2-1}\,dy$$

$$\begin{array}{r} 1 + \frac{1}{y^2-1} \\ y^2-1\,\overline{)\,y^2} \\ -(y^2-1) \\ \hline 1 \end{array}$$

$$= \int_{\sqrt2}^{\sqrt{1+e^2}} \left[1 + \frac{1}{(y+1)(y-1)}\right]dy = \int_{\sqrt2}^{\sqrt{1+e^2}} \left[1 - \frac12\cdot\frac{1}{y+1} + \frac12\cdot\frac{1}{y-1}\right]dy$$

$\frac{A}{y+1} + \frac{B}{y-1}$

$$= \left[y - \frac12\ln|y+1| + \frac12\ln|y-1|\right]_{\sqrt2}^{\sqrt{1+e^2}}$$

$1 = A(y-1) + B(y+1)$

$y=1: \ 1 = 2B \quad \boxed{B = \frac12}$

$y=-1: \ 1 = -2A \quad \boxed{A = -\frac12}$

$$= \sqrt{1+e^2} - \frac12\ln\left|\sqrt{1+e^2}+1\right| + \frac12\ln\left|\sqrt{1+e^2}-1\right| - \left[\sqrt2 \overbrace{- \frac12\ln|\sqrt2+1| + \frac12\ln|\sqrt2-1|}^{\frac12\ln\left|\frac{\sqrt2-1}{\sqrt2+1}\right|}\right]$$

$$= \boxed{\sqrt{1+e^2} + \frac12\ln\left|\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right| - \sqrt2 + \frac12\ln\left|\frac{\sqrt2-1}{\sqrt2+1}\right|}$$

to match book's answer ↗

$$= \sqrt{1+e^2} + \frac12\ln\left|\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\cdot\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}-1}\right| - \sqrt2 - \frac12\ln\left|\frac{\sqrt2-1}{\sqrt2+1}\cdot\frac{\sqrt2-1}{\sqrt2-1}\right|$$

$$= \sqrt{1+e^2} + \frac12\ln\left|\frac{(\sqrt{1+e^2}-1)^2}{\cancel{1}+e^2\cancel{-1}}\right| - \sqrt2 - \frac12\ln\left|\frac{(\sqrt2-1)^2}{2-1}\right|$$

$$= \sqrt{1+e^2} + \frac12\ln\left(\sqrt{1+e^2}-1\right)^2 - \frac12\ln\left(e^2\right) - \sqrt2 - \frac12\ln\left(\sqrt2-1\right)^2 = \sqrt{1+e^2} + \ln\left(\sqrt{1+e^2}-1\right) - 1 \to -\sqrt2 - \ln\left(\sqrt2-1\right)$$

3. $y = \cos x, \quad 0 \le x \le 2\pi$

$y' = -\sin x, \quad (y')^2 = \sin^2 x.$ $\quad L = \int_0^{2\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \boxed{\int_0^{2\pi} \sqrt{1+\sin^2 x}\, dx}$

5. $x = y + y^3, \quad 1 \le y \le 4$

$\frac{dx}{dy} = 1 + 3y^2.$ $\quad \left|\frac{dx}{dy}\right|^2 = \left(1+3y^2\right)^2 = 1 + 6y^2 + 9y^4.$

$L = \int_1^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^4 \sqrt{1 + 1 + 6y^2 + 9y^4}\, dy = \boxed{\int_1^4 \sqrt{2 + 6y^2 + 9y^4}\, dy}$