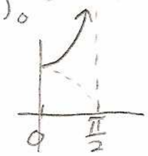
7.8 homework

1. a. $\int_1^{\infty} x^4 e^{-x^4}\,dx$ $\boxed{\text{Type 1}}$ b. $\int_0^{\frac{\pi}{2}} \sec x\,dx$ V.A. at $x=\frac{\pi}{2}$ $\boxed{\text{Type 2}}$



c. $\int_0^2 \frac{x\,dx}{(x-3)(x-2)}$

V.A. at $x=2$ $\boxed{\text{Type 2}}$

d. $\int_{-\infty}^{0} \frac{1}{x^2+5}\,dx$ $\boxed{\text{Type 1}}$

If the limit equals $\pm\infty$ or DNE $\Rightarrow$ $\boxed{\text{Diverges}}$

If the limit equals a finite # $\Rightarrow$ $\boxed{\text{Converges}}$

Type 1 Improper Integral: Limits of integration involve $-\infty$ and/or ∞.

Type 2 Improper Integral: The integrand is discontinuous somewhere on $[a,b]$.

3. $\int_1^t x^{-3}\,dx = \left[\frac{x^{-2}}{-2}\right]_1^t = -\frac{1}{2}\left[\frac{1}{x^2}\right]_1^t = -\frac{1}{2}\left[\frac{1}{t^2}-1\right] = \frac{1}{2}\left[1-\frac{1}{t^2}\right]$

t	Area
10	$\frac{1}{2}\left[1-\frac{1}{100}\right] = \frac{1}{2}\cdot\frac{99}{100} = \frac{99}{200}$.
100	$\frac{1}{2}\left[1-\frac{1}{10{,}000}\right] = \frac{1}{2}\cdot\frac{9999}{10{,}000} = \frac{9999}{20{,}000}$
1000	$\frac{1}{2}\left[1-\frac{1}{1{,}000{,}000}\right] = \frac{1}{2}\cdot\frac{999{,}999}{1{,}000{,}000} = \frac{999{,}999}{2{,}000{,}000}$

Total Area $= \int_1^{\infty} x^{-3}\,dx = \lim_{t\to\infty}\int_1^t x^{-3}\,dx$

$= \lim_{t\to\infty} \frac{1}{2}\left[1-\frac{1}{t^2}\right]$

$= \boxed{\frac{1}{2}}$.

5. $\int_1^{\infty} \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty} -\frac{1}{3}\cdot\left[\frac{1}{3x+1}\right]_1^t = -\frac{1}{3}\cdot\lim_{t\to\infty}\left[\overset{0}{\frac{1}{3t+1}} - \frac{1}{4}\right]$

$= -\frac{1}{3}\cdot\left[-\frac{1}{4}\right] = \boxed{\frac{1}{12}}$

$\boxed{\text{Convergent}}$

$\frac{1}{3}\int \frac{1\cdot 3}{(3x+1)^2}\,dx = \frac{1}{3}\int u^{-2}\,du = \frac{1}{3}\cdot\frac{-1}{u} = -\frac{1}{3}\cdot\frac{1}{u} = -\frac{1}{3}\cdot\frac{1}{3x+1}$

$u=3x+1$, $du=3\,dx$

7. $\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1}\frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1}$

$= -2\cdot\lim_{t\to-\infty}\left[\sqrt{3}-\sqrt{2-t}\right]$

$= -2\cdot(-\infty) = \infty$ $\boxed{\text{Divergent}}$

$-\int \frac{1\,(-)}{\sqrt{2-w}}\,dw = -\int u^{-\frac{1}{2}}\,du = -2u^{\frac{1}{2}} = -2\sqrt{2-w}$

$u=2-w$

$du=-dw$

9. $\int_4^{\infty} e^{-\frac{1}{2}y}\,dy = \lim_{t\to\infty}\int_4^{t} e^{-\frac{1}{2}y}\,dy = \lim_{t\to\infty}\left[-2e^{-\frac{1}{2}y}\right]_4^t = -2\cdot\lim_{t\to\infty}\left[\cancel{e^{-\frac{1}{2}t}}^{\to 0} - e^{-2}\right]$

$-2\int e^{-\frac{1}{2}y}\,dy\left(-\frac{1}{2}\right) = -2\int e^{u}\,du = -2e^{u} = -2e^{-\frac{1}{2}y}$

$= -2\cdot\left[\frac{-1}{e^2}\right] = \boxed{\frac{2}{e^2}}$ $\boxed{\text{Convergent}}$

$u = -\frac{1}{2}y$

$du = -\frac{1}{2}dy$

11. $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x}{1+x^2}\,dx.$

$\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{x}{1+x^2}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}\ln|1+x^2|\right]_t^0 = \frac{1}{2}\cdot\lim_{t\to-\infty}\left[0-\ln|1+t^2|\right]$

$= \frac{1}{2}\cdot\left[-\infty\right] = -\infty.$

$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2|$

$u = 1+x^2$

$du = 2x\,dx$

Since the 1st integral is Divergent, there is no need to calculate the 2nd integral. The original integral is $\boxed{\text{Divergent}}$.

13. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx.$ $\left[-\frac{1}{2}\int xe^{-x^2}\,dx\,(-2) \quad u=-x^2 \quad du=-2x\,dx \quad = -\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{-x^2}.\right]$

$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{t\to-\infty}\int_{t}^{0} xe^{-x^2}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_t^0 = -\frac{1}{2}\cdot\lim_{t\to-\infty}\left[1-\cancel{\frac{1}{e^{t^2}}}^{\to 0}\right] = -\frac{1}{2}.$

$\int_{0}^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_{0}^{t} xe^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^t = -\frac{1}{2}\cdot\lim_{t\to\infty}\left[\cancel{\frac{1}{e^{t^2}}}^{\to 0}-1\right] = -\frac{1}{2}(-1) = \frac{1}{2}.$

$-\frac{1}{2}+\frac{1}{2} = \boxed{0}$ $\boxed{\text{Convergent}}$

15. $\int_{2\pi}^{\infty} \sin\theta\,d\theta = \lim_{t\to\infty}\int_{2\pi}^{t} \sin\theta\,d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = -\lim_{t\to\infty}\left[\cos t - 1\right] =$ Does Not Exist (DNE)

$\boxed{\text{Divergent}}$

17. $\int_1^\infty \frac{x+1}{x^2+2x}\,dx = \lim_{t\to\infty}\int_1^t \frac{x+1}{x^2+2x}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\ln|x^2+2x|\right]_1^t = \frac{1}{2}\cdot\lim_{t\to\infty}\left[\ln|t^2+2t| - \ln 3\right]$

$= \frac{1}{2}\cdot\infty = \infty.$ $\boxed{\text{Divergent}}$

$\frac{1}{2}\int \frac{x+1}{x^2+2x}(2)\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|x^2+2x|$

$u = x^2+2x$
$du = 2x+2\,dx$
$du = 2(x+1)\,dx$

19. $\int_0^\infty s e^{-5s}\,ds = \lim_{t\to\infty}\int_0^t s e^{-5s}\,ds = \lim_{t\to\infty}\left[-\frac{1}{5}se^{-5s} - \frac{1}{25}e^{-5s}\right]_0^t = \lim_{t\to\infty}\left[-\frac{1}{25}e^{-5s}(5s+1)\right]_0^t$

$\int s e^{-5s}\,ds = -\frac{1}{5}se^{-5s} + \frac{1}{5}\int e^{-5s}\,ds = -\frac{1}{5}se^{-5s} - \frac{1}{25}e^{-5s}$

$u = s \quad v = -\frac{1}{5}e^{-5s}$
$du = ds \quad dv = e^{-5s}\,ds$

L'Hospital's Rule

$= -\frac{1}{25}\lim_{t\to\infty}\left[e^{-5t}(5t+1) - 1\right] = -\frac{1}{25}\lim_{t\to\infty}\left[\frac{5t-1}{e^{5t}} - 1\right] \overset{H}{=} -\frac{1}{25}\left[\left(\lim_{t\to\infty}\frac{5}{5e^{5t}}\right) - 1\right]$

$= -\frac{1}{25}\left[0-1\right] = \boxed{\frac{1}{25}}$ $\boxed{\text{Convergent}}$

21. $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[(\ln t)^2 - 0\right] = \infty.$ $\boxed{\text{Divergent}}$

$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$

$u = \ln x$
$du = \frac{1}{x}\,du$

23. $\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^\infty \frac{x^2}{9+x^6}\,dx.$

$\frac{1}{3}\int \frac{3x^2}{9+x^6}\,dx = \frac{1}{3}\int \frac{1}{9+u^2}\,du = \frac{1}{3}\cdot\frac{1}{3}\arctan\frac{u}{3}$
$= \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right).$

$u = x^3$
$du = 3x^2$

$\int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx = \lim_{t\to-\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_t^0 = \frac{1}{9}\lim_{t\to-\infty}\left[0 - \tan^{-1}\left(\frac{t^3}{3}\right)\right] = \frac{1}{9}\left[0 - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{18}.$

$\int_0^\infty \frac{x^2}{9+x^6}\,dx = \lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t = \frac{1}{9}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t^3}{3}\right) - 0\right] = \frac{1}{9}\left[\frac{\pi}{2}\right] = \frac{\pi}{18}.$

$\therefore \frac{\pi}{18} + \frac{\pi}{18} = \boxed{\frac{\pi}{9}}$ $\boxed{\text{Convergent}}$

25. $\int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2(\ln x)^2}\right]_e^t = -\frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{(\ln t)^2}-1\right] = -\frac{1}{2}[0-1] = \boxed{\frac{1}{2}}$ $\boxed{\text{Convergent}}$

$\int \frac{1}{x(\ln x)^3}\,dx = \int u^{-3}\,du = \frac{u^{-2}}{-2} = -\frac{1}{2(\ln x)^2}$

$u = \ln x$
$du = \frac{1}{x}dx$

27. $\int_0^1 \frac{3}{x^5}\,dx = \lim_{t\to 0^+}\int_t^1 3x^{-5}\,dx = \lim_{t\to 0^+}\left[\frac{3x^{-4}}{-4}\right]_t^1 = -\frac{3}{4}\lim_{t\to 0^+}\left[\frac{1}{x^4}\right]_t^1$

$= -\frac{3}{4}\lim_{t\to 0^+}\left[1-\frac{1}{t^4}\right] = -\frac{3}{4}[1-\infty] = -\frac{3}{4}(-\infty) = \infty.$ $\boxed{\text{Divergent}}$

29. $\int_{-2}^{14}\frac{dx}{\sqrt[4]{x+2}} = \lim_{t\to -2^+}\int_t^{14}\frac{dx}{\sqrt[4]{x+2}} = \frac{4}{3}\lim_{t\to -2^+}\left[(x+2)^{\frac{3}{4}}\right]_t^{14} = \frac{4}{3}\lim_{t\to -2^+}\left[8+(t+2)^{\frac{3}{4}}\right]$

$= \frac{4}{3}[8+0] = \boxed{\frac{32}{3}}$ $\boxed{\text{Convergent}}$

$\int \frac{dx}{\sqrt[4]{x+2}} = \int u^{-\frac{1}{4}}\,du = \frac{4}{3}u^{\frac{3}{4}} = \frac{4}{3}(x+2)^{\frac{3}{4}}$

$u = x+2$
$du = dx$

31. $\int_{-2}^{3}\frac{1}{x^4}\,dx = \int_{-2}^{0}x^{-4}\,dx + \int_0^3 x^{-4}\,dx.$ $\int x^{-4}\,dx = \frac{x^{-3}}{-3} = -\frac{1}{3x^3}$

$\int_{-2}^{0}x^{-4}\,dx = \lim_{t\to 0^-}\left[-\frac{1}{3x^3}\right]_{-2}^{t} = -\frac{1}{3}\lim_{t\to 0^-}\left[\frac{1}{t^3}-\left(-\frac{1}{8}\right)\right] = -\frac{1}{3}\left[-\infty+\frac{1}{8}\right] = -\frac{1}{3}(-\infty)$
$= \infty.$

Since the 1st integral diverges, there's no need to do the 2nd integral; the original integral is $\boxed{\text{Divergent}}$.

33. $\int_0^{33}(x-1)^{-\frac{1}{5}}\,dx = \int_0^1 (x-1)^{-\frac{1}{5}}\,dx + \int_1^{33}(x-1)^{-\frac{1}{5}}\,dx.$ $\int \frac{1}{(x-1)^{\frac{1}{5}}}\,dx = \int u^{-\frac{1}{5}}\,du = \frac{5}{4}u^{\frac{4}{5}} = \frac{5}{4}(x-1)^{\frac{4}{5}}$

$u = x-1,\ du = dx$

$\int_0^1 (x-1)^{-\frac{1}{5}}\,dx = \lim_{t\to 1^-}\left[\frac{5}{4}(x-1)^{\frac{4}{5}}\right]_0^t = \frac{5}{4}\lim_{t\to 1^-}\left[\cancelto{0}{(t-1)^{\frac{4}{5}}}-1\right] = \frac{5}{4}[0-1] = -\frac{5}{4}.$

$\int_1^{33}(x-1)^{-\frac{1}{5}}\,dx = \lim_{t\to 1^+}\left[\frac{5}{4}(x-1)^{\frac{4}{5}}\right]_t^{33} = \frac{5}{4}\lim_{t\to 1^+}\left[16-\cancelto{0}{(t-1)^{\frac{4}{5}}}\right] = \frac{5}{4}[16-0] = 20.$

$-\frac{5}{4}+\frac{20}{1} = -\frac{5}{4}+\frac{80}{4} = \boxed{\frac{75}{4}}$ $\boxed{\text{Convergent}}$

35. $\displaystyle\int_0^3 \frac{dx}{x^2-6x+5} = \int_0^1 \frac{dx}{(x-5)(x-1)} + \int_1^3 \frac{dx}{(x-5)(x-1)}$

$(x-5)(x-1)$

$\displaystyle\int \frac{dx}{(x-5)(x-1)} = \int\left(\frac{A}{x-5}+\frac{B}{x-1}\right)dx$ $\quad 1 = A(x-1)+B(x-5)$

$1 = (A+B)x + (-A-5B)$

$A+B=0$
$-A-5B=1$
$-4B=1$
$B=-\frac{1}{4}$. $A=\frac{1}{4}$.

$= \frac{1}{4}\ln|x-5| - \frac{1}{4}\ln|x-1|$.

$\displaystyle\int_0^1 \frac{dx}{(x-5)(x-1)} = \lim_{t\to1^-}\left[\frac{1}{4}\ln|x-5|-\frac{1}{4}\ln|x-1|\right]_0^t = \frac{1}{4}\lim_{t\to1^-}\left[(\ln|t-5|-\ln|t-1|)-(\ln 5-0)\right]$

$= \frac{1}{4}\left[\ln 4 - \ln(0^+) - \ln 5\right] = \frac{1}{4}\left[\ln 4-(-\infty)-\ln 5\right] = \infty$. Since I_1 diverges, no need to do I_2.

The original integral is $\boxed{\text{divergent.}}$

37. $\displaystyle\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}dx = \lim_{t\to0^-}\int_{-1}^t \frac{e^{\frac{1}{x}}}{x^3}dx.$

$\displaystyle-\int \frac{e^{\frac{1}{x}}}{x^3}dx = -\int ue^u\,du = -\left[ue^u - \int e^u\,du\right] = -ue^u + e^u = e^u(1-u) = e^{\frac{1}{x}}\left(1-\frac{1}{x}\right)$

$u = \frac{1}{x}$ $\quad U = u \quad V = e^u$

$du = -\frac{1}{x^2}dx$ $\quad dU = du \quad dV = e^u du$

$\displaystyle\lim_{t\to0^-}\int_{-1}^t \frac{e^{\frac{1}{x}}}{x^3}dx = \lim_{t\to0^-}\left[e^{\frac{1}{x}}\left(1-\frac{1}{x}\right)\right]_{-1}^t = \lim_{t\to0^-}\left[\underbrace{e^{\frac{1}{t}}\left(1-\frac{1}{t}\right)}_{0\cdot\infty \text{ Need L'Hospital's Rule}} - e^{-1}(1+1)\right]$

$\displaystyle= \lim_{t\to0^-}\left[\frac{1-\frac{1}{t}}{e^{-\frac{1}{t}}}\right] - 2e^{-1} \overset{H}{=} \lim_{t\to0^-}\left[\frac{\cancel{\frac{1}{t^2}}}{e^{-\frac{1}{t}}\cdot\cancel{\frac{1}{t^2}}}\right] - \frac{2}{e} = \lim_{t\to0^-}\left(e^{\frac{1}{t}}\right)^{\nearrow 0} - \frac{2}{e} = \boxed{-\frac{2}{e}}$

$\boxed{\text{Convergent}}$

39. $\int_0^2 z^2 \ln z\, dz = \lim_{t\to 0^+} \int_t^2 z^2 \ln z\, dz = \lim_{t\to 0^+} \left[\frac{1}{9} z^3(3\ln z - 1)\right]_t^2$

$\int z^2 \ln z\, dz = \frac{1}{3}z^3 \ln z - \int \frac{1}{3}z^2\, dz = \frac{1}{3}z^3\ln z - \frac{1}{9}z^3 = \frac{1}{9}z^3(3\ln z - 1)$

$u = \ln z \quad v = \frac{1}{3}z^3$

$du = \frac{1}{z}dz \quad dv = z^2 dz$

$-\frac{\infty}{\infty}$ ready for L'Hospital

$= \frac{1}{9}\lim_{t\to 0^+}\left[8(3\ln 2 - 1) - \underbrace{t^3(3\ln t - 1)}_{\substack{0\cdot(-\infty)\\ \text{Need L'Hospital}}}\right] = \frac{8}{9}(3\ln 2 - 1) - \frac{1}{9}\lim_{t\to 0^+}\left[\frac{3\ln t - 1}{t^{-3}}\right]$

$\overset{H}{=} \frac{8}{9}(3\ln 2 - 1) - \frac{1}{9}\lim_{t\to 0^+}\left[\frac{\frac{3}{t}}{-\frac{3}{t^4}}\right] = \frac{8}{9}(3\ln 2 - 1) - \frac{1}{9}\lim_{t\to 0^+}\left[\cancel{-t^3}\right]^{\to 0} = \boxed{\frac{8}{9}(3\ln 2 - 1)}$ $\boxed{\text{Convergent}}$

49. $\int_0^\infty \frac{x}{x^3+1} = \int_0^1 \frac{x}{x^3+1}dx + \int_1^\infty \frac{x}{x^3+1}dx \rightarrow \frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}$

$\le \underbrace{\int_0^1 \frac{x}{x^3+1}dx}_{\substack{\text{a constant}\\ \text{(converges)}}} + \underbrace{\int_1^\infty \frac{1}{x^2}dx}_{}$

Converges because $p=2$, and $\int_1^\infty \frac{1}{x^p}dx$ converges for $p>1$. (Theorem 2 P. 511)

Therefore, $\int_0^\infty \frac{x}{x^3+1}dx$ is $\boxed{\text{convergent}}$ by the Comparison Theorem.

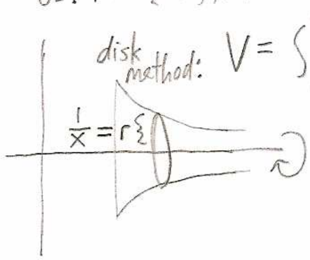
51. $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}dx$. $\quad \frac{x+1}{\sqrt{x^4-x}} > \frac{x+1}{\sqrt{x^4}} = \frac{x+1}{x^2} > \frac{x}{x^2} = \frac{1}{x}$, and $\int_1^\infty \frac{1}{x}dx$ diverges $(p=1)$

(Thrm. 2 P. 511 says $\int_1^\infty \frac{1}{x^p}dx$ diverges for $p\le 1$)

so $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}dx$ $\boxed{\text{diverges.}}$

63. $R = \{(x,y) \mid x \ge 1,\ 0 \le y \le \frac{1}{x}\}$ about x axis.

disk method: $V = \int_1^\infty A(x)\,dx = \int_1^\infty \pi\cdot\frac{1}{x^2}dx = \pi\lim_{t\to\infty}\int_1^t x^{-2}dx = \pi\lim_{t\to\infty}\left[\frac{-1}{x}\right]_1^t$



$A(x) = \pi r^2 = \pi\cdot\left(\frac{1}{x}\right)^2 = \pi\cdot\frac{1}{x^2}$

$= \pi\lim_{t\to\infty}\left[-\cancel{\frac{1}{t}}^{\,0} + 1\right] = \pi[0+1] = \boxed{\pi}$ The volume is finite.