7.5 homework

1. $\int \cos x(1+\sin^2 x)\,dx = \int (1+u^2)\,du = u+\frac{1}{3}u^3+C = \boxed{\sin x+\frac{1}{3}\sin^3 x+C}$

$u=\sin x,\ du=\cos x\,dx$

3. $\int \frac{\sin x+\sec x}{\tan x}\,dx = \int \frac{\left(\sin x+\frac{1}{\cos x}\right)\cdot\cos x}{\frac{\sin x}{\cos x}\cdot\cos x}\,dx = \int \frac{\sin x\cos x+1}{\sin x}\,dx = \int \cos x\,dx+\int \csc x\,dx$

$= \boxed{\sin x+\ln|\csc x-\cot x|+C}$

5. $\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du = \int_{-3}^{-1} \frac{2u+6}{u^2}\,du = 2\int_{-3}^{-1} \frac{1}{u}\,du + 6\int_{-3}^{-1} u^{-2}\,du$

$u=t-3 \to u+3=t$

$du=dt$

$= 2\Big[\ln|u|\Big]_{-3}^{-1} + 6\left[\frac{u^{-1}}{-1}\right]_{-3}^{-1} = 2[0-\ln 3] - 6\left[\frac{1}{u}\right]_{-3}^{-1}$

$= -2\ln 3 - 6\left[-1-\left(-\frac{1}{3}\right)\right] = -2\ln 3+6-2 = \boxed{-2\ln 3+4}$

7. $\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^u\,du = e^u\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \boxed{e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}}}$

$u=\arctan y$

$du=\frac{1}{1+y^2}\,dy$

9. $\int_1^3 r^4\ln r\,dr = \left[\frac{r^5}{5}\ln r\right]_1^3 - \int_1^3 \frac{1}{5}r^4\,dr = \left(\frac{243}{5}\ln 3-0\right) - \frac{1}{5}\left[\frac{r^5}{5}\right]_1^3$

$= \frac{243}{5}\ln 3 - \frac{1}{25}(243-1) = \boxed{\frac{243}{5}\ln 3-\frac{242}{25}}$

$u=\ln r \qquad v=\frac{r^5}{5}$

$du=\frac{1}{r}\,dr \qquad dv=r^4\,dr$

11. $\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{x-1}{(x-2)^2+1}\,dx = \int \frac{u+2-1}{u^2+1}\,du = \int \frac{u+1}{u^2+1}\,du = \frac{1}{2}\int \frac{2u}{u^2+1}\,du + \int \frac{du}{u^2+1}$

$y=u^2+1$

$dy=2u\,du$

$x^2-4x+4-4+5$

$(x-2)^2+1$

$u=x-2 \to u+2=x$

$du=dx$

$= \frac{1}{2}\int \frac{1}{y}\,dy + \tan^{-1}u+C = \frac{1}{2}\ln(u^2+1)+\tan^{-1}u+C$

$= \frac{1}{2}\ln\left((x-2)^2+1\right)+\tan^{-1}(x-2)+C$

$= \boxed{\frac{1}{2}\ln\left(x^2-4x+5\right)+\tan^{-1}(x-2)+C}$

OR:

$\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{x^2-4x+5}\,dx = \frac{1}{2}\int \frac{2[(x-2)+1]}{x^2-4x+5}\,dx$

$u=x^2-4x+5$

$du=2x-4\,dx$

$du=2(x-2)\,dx$

$= \frac{1}{2}\int \frac{2(x-2)}{x^2-4x+5}\,dx + \int \frac{1}{x^2-4x+5}\,dx = \frac{1}{2}\int \frac{1}{u}\,du + \int \frac{1}{(x-2)^2+1}$

$= \frac{1}{2}\ln|u| + \tan^{-1}(x-2)+C = \boxed{\frac{1}{2}\ln(x^2-4x+5)+\tan^{-1}(x-2)+C}$

13. $\int \sin^3\theta \cos^5\theta \, d\theta = -\int (1-\cos^2\theta)\cos^5\theta \overset{(-1)}{\sin\theta \, d\theta} = -\int (1-u^2)u^5 \, du = -\int (u^5-u^7)du$

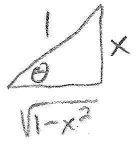
$u=\cos\theta, \; du = -\sin\theta \, d\theta$

$= -\left[\frac{u^6}{6} - \frac{u^8}{8}\right] + C = \boxed{\frac{1}{8}\cos^8\theta - \frac{1}{6}\cos^6\theta + C}$

15. $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \int \frac{\cos\theta \, d\theta}{\cos^3\theta} = \int \sec^2\theta \, d\theta = \tan\theta + C = \boxed{\frac{x}{\sqrt{1-x^2}} + C}$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$



$\left(\sqrt{1-x^2}\right)^3 = \left(\sqrt{1-\sin^2\theta}\right)^3 = \left(\sqrt{\cos^2\theta}\right)^3 = \cos^3\theta$

17. $\int x \sin^2 x \, dx = \int x \cdot \frac{1}{2}(1-\cos 2x)\,dx = \int x\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x \, dx$

$u = x \quad v = \frac{\sin 2x}{2}$

$du = dx \quad dv = \cos 2x \, dx$

$= \frac{1}{2}\frac{x^2}{2} - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x \, dx\right]$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x + \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] = \boxed{\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C}$

OR:

$\int x \sin^2 x \, dx = x\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) - \int\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)dx = \frac{1}{2}x^2 - \frac{1}{4}x\sin 2x - \int \frac{1}{2}x\,dx + \frac{1}{4}\int \sin 2x \, dx$

$v = \frac{1}{2}x - \frac{1}{4}\sin 2x$

$u = x$

$du = dx \quad dv = \sin^2 x \, dx = \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

$= \frac{1}{2}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{4}\cdot\left[\frac{-\cos 2x}{2}\right] + C$

$= \frac{1}{2}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{4}x^2 - \frac{1}{8}\cos 2x + C$

$= \boxed{\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C}$

19. $\int e^{(x+e^x)}dx = \int e^x \cdot e^{e^x} dx = \int e^u du = e^u + C = \boxed{e^{e^x} + C}$

$u = e^x; \; du = e^x dx$

Dead end this way → 21. ~~$\int \arctan\sqrt{x}\,dx = x\arctan\sqrt{x} - \int x \cdot \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}dx = x\arctan x - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}dx$~~ ← deadend

~~Parts: $u = \arctan\sqrt{x} \quad v = x$~~ ~~$= \arctan x - 2\int \frac{\sqrt{u-1}}{u}du$~~

~~$du = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}dx \quad dv = dx$~~

$u = 1+x$

$u-1 = x$

$\sqrt{u-1} = \sqrt{x}$

$du = dx$

works this way → 21. $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2\sqrt{x}\,du = dx$

$2u\,du = dx$

$\int \arctan\sqrt{x}\,dx = \int \arctan u \cdot 2u\,du = 2\int \tan^{-1}u\,(u\,du)$

$U = \tan^{-1}u \quad V = \frac{u^2}{2}$

$dU = \frac{1}{1+u^2}du \quad dV = u\,du$

$= 2\left[\frac{1}{2}u^2\tan^{-1}u - \int \frac{1}{2}\cdot\frac{u^2}{u^2+1}du\right]$

$$\begin{array}{r} 1 - \frac{1}{u^2+1} \\ u^2+1 \,\overline{)\, u^2 \quad} \\ -(u^2+1) \\ \hline -1 \end{array}$$

$= u^2\tan^{-1}u - \int\left[1 - \frac{1}{u^2+1}\right]du = \boxed{x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C}$

7.5 homework continued

23. $\int_0^1 (1+\sqrt{x})^8\,dx = \int_1^2 u^8 \cdot 2(u-1)\,du = 2\int_1^2 (u^9-u^8)\,du = 2\left[\frac{u^{10}}{10}-\frac{u^9}{9}\right]_1^2$

$u = 1+\sqrt{x} \rightarrow u-1 = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2\sqrt{x}\,du = dx$

$2(u-1)\,du = dx$

$= 2\left[\left(\frac{1024}{10}-\frac{512}{9}\right)-\left(\frac{1}{10}-\frac{1}{9}\right)\right] = 2\left[\frac{1023}{10}-\frac{511}{9}\right] = 2\left[\frac{9207-5110}{90}\right]$

$\begin{array}{r} 1023 \\ \times\ 9 \\ \hline 9207 \end{array} \quad \begin{array}{r} 9207 \\ -5110 \\ \hline 4097 \end{array}$

$= 2\left[\frac{4097}{90}\right] = \boxed{\frac{4097}{45}}$

25. $\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int 3\,dx + \int \frac{6x+22}{x^2-2x-8}\,dx = 3x + \int\left[\frac{A}{x-4}+\frac{B}{x+2}\right]dx$

$(x-4)(x+2)$

$$\begin{array}{r} 3 + \frac{6x+22}{x^2-2x-8} \\ x^2-2x-8\ \overline{)\ 3x^2+0x-2} \\ -(3x^2-6x-24) \\ \hline 6x+22 \end{array}$$

$6x+22 = A(x+2)+B(x-4)$

$x=4:\ 46 = 6A \rightarrow A = \frac{23}{3}.$

$x=-2:\ 10 = -6B \rightarrow B = -\frac{5}{3}.$

$= \boxed{3x + \frac{23}{3}\ln|x-4| - \frac{5}{3}\ln|x+2| + C.}$

Exotic substitution trying for arctan (didn't get arctan, but still worked) (took too long, though)

27. $\int \frac{dx}{1+e^x} = \int \frac{\frac{2}{u}\,du}{1+u^2} = 2\int \frac{1}{u(u^2+1)}\,du = 2\int\left[\frac{A}{u}+\frac{Bu+C}{u^2+1}\right]$

$1 = A(u^2+1)+(Bu+C)u$

$1 = Au^2+A+Bu^2+Cu$

$1 = (A+B)u^2+Cu+A$

$\boxed{C=0}$ $\boxed{A=1}$ $\boxed{B=-1}$

$u^2 = e^x \rightarrow 2u\,du = e^x dx \rightarrow \frac{2u\,du}{e^x} = dx \rightarrow \frac{2u\,du}{u^2} = dx \rightarrow \frac{2}{u}\,du = dx$

$= 2\int \frac{1}{u}\,du + 2\int \frac{-u}{u^2+1}\,du = 2\ln|u| - 2\cdot\frac{1}{2}\int\frac{2\cdot u}{u^2+1}\,du = 2\ln|e^{\frac{x}{2}}| - \ln|u^2+1| + C$

$u = \sqrt{e^x} = (e^x)^{\frac{1}{2}} = e^{\frac{x}{2}}$

$= 2\ln(e^{\frac{x}{2}}) - \ln(e^x+1)+C.$

$= \ln(e^{\frac{x}{2}})^2 - \ln(e^x+1)+C$

$= \ln(e^x) - \ln(e^x+1)+C$

$= \boxed{x - \ln(e^x+1)+C}$

Basic, effective

OR ⑰ wait

OR (27.) $\int \frac{dx}{1+e^x} = \int \frac{\left(\frac{du}{u-1}\right)}{u} = \int \frac{1}{u(u-1)}\,du$

$u = 1+e^x \rightarrow u-1 = e^x$

$du = e^x dx = (u-1)dx$

$\frac{du}{u-1} = dx$

$= \int\left[\frac{A}{u}+\frac{B}{u-1}\right]du$

$1 = A(u-1)+Bu$

$1 = (A+B)u - A$

$\boxed{A=-1}$

$\boxed{B=1}$

$= -\int\frac{1}{u}\,du + \int\frac{1}{u-1}\,du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln(1+e^x-1)+C$

$= -\ln(1+e^x) + \ln(e^x) + C$

$= \boxed{-\ln(1+e^x)+x+C}$

OR 27. $\int \frac{dx}{(1+e^x)}\frac{(e^{-x})}{(e^{-x})} = -\int \frac{-e^{-x}}{e^{-x}+1}\,dx = -\int \frac{du}{u} = -\ln|u|+C = -\ln|e^{-x}+1|+C$

creative attack $\nearrow$

$u = e^{-x}+1$

$du = -e^{-x}\,dx$

$= -\ln\left|\frac{1}{e^x}+1\right|+C = -\ln\left|\frac{1+e^x}{e^x}\right|+C$

$= -\left[\ln(1+e^x)-\ln(e^x)\right] = \boxed{-\ln(1+e^x)+x+C}$

29. $\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5 \left[3-\frac{7}{w+2}\right]dw = \left[3w\right]_0^5 - 7\left[\ln|w+2|\right]_0^5$

$$\begin{array}{r} 3-\frac{7}{w+2} \\ w+2\overline{)\,3w-1} \\ -(3w+6) \\ \hline -7 \end{array}$$

$= 15-7\left[\ln 7-\ln 2\right] = \boxed{15-7\ln\frac{7}{2}}$ OR $\boxed{15+7\ln\frac{2}{7}}$

31. $\int \sqrt{\frac{(1+x)}{(1-x)}\cdot\frac{(1+x)}{(1+x)}} = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx + -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = \sin^{-1}x - \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$u = 1-x^2$

$du = -2x\,dx$

$= \sin^{-1}x - \frac{1}{2}\cdot 2u^{\frac{1}{2}}+C = \boxed{\sin^{-1}x-\sqrt{1-x^2}+C.}$

33. $\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \int 2\cos\theta\cdot 2\cos\theta\,d\theta = 4\int \cos^2\theta\,d\theta$

$-x^2-2x+3$

$-(x^2+2x)+3$

$-(x^2+2x+1-1)+3$

$-(x+1)^2+1+3$

$-(x+1)^2+4$

$4-(x+1)^2$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\sqrt{4-(x+1)^2} = \sqrt{4-4\sin^2\theta}$

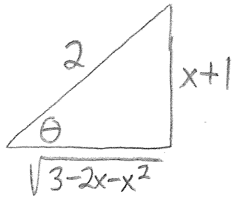
$= \sqrt{4(1-\sin^2\theta)}$

$= 2\cos\theta$

$= 4\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$

$2\sin\theta\cos\theta$

$= 2\left[\theta+\frac{\sin 2\theta}{2}\right]+C$



$= 2\left[\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{\cancel{2}}\cdot\cancel{2}\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3-2x-x^2}}{2}\right]+C$

$= \boxed{2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+C.}$

35. $\int_{-1}^{1} x^8\sin x\,dx = -x^8\cos x\Big]_{-1}^{1} + 8\int_{-1}^{1} x^7\cos x\,dx$ = etc. could keep going, but not necessary.

$u = x^8$ $\quad v = -\cos x$

$du = 8x^7\,dx$ $\quad dv = \sin x\,dx$

The answer is $\boxed{0}$ since the integrand is an odd function and $\int_{-a}^{a}$ odd function $= 0$.

37. $\int_0^{\frac{\pi}{4}} \cos^2\theta\tan^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\cdot\frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)-\left(0-\frac{1}{2}\sin 0\right)\right] = \frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{2}\right] = \boxed{\frac{\pi}{8}-\frac{1}{4}}$

39. $$\int \frac{\sec\theta\tan\theta}{\sec^2\theta-\sec\theta}\,d\theta = \int \frac{\left[\frac{1}{\cancel{\cos\theta}}\cdot\frac{\sin\theta}{\cancel{\cos\theta}}\right]\cdot\cancel{\cos^2\theta}}{\left[\frac{1}{\cos^2\theta}-\frac{1}{\cos\theta}\right]\cdot\cos^2\theta} = \int\frac{\sin\theta}{1-\cos\theta}\,d\theta = \int\frac{1}{u}\,du = \ln|u|+C$$

$$= \boxed{\ln|1-\cos\theta|+C}$$

$u = 1-\cos\theta$

$du = \sin\theta\,d\theta$

41. $$\int\theta\tan^2\theta\,d\theta = \theta(\tan\theta-\theta)-\int(\tan\theta-\theta)\,d\theta = \theta\tan\theta-\theta^2-\ln|\sec\theta|+\tfrac{1}{2}\theta^2+C$$

$$= \boxed{\theta\tan\theta-\tfrac{1}{2}\theta^2-\ln|\sec\theta|+C}$$

OR $\boxed{\theta\tan\theta-\tfrac{1}{2}\theta^2+\ln|\cos\theta|+C}$

$u=\theta$ $\quad v=\tan\theta-\theta$

$du=d\theta$ $\quad dv=\tan^2\theta\,d\theta$

$\quad =(\sec^2\theta-1)\,d\theta$

OR 41. $$\int\theta\tan^2\theta\,d\theta = \int\theta(\sec^2\theta-1)\,d\theta = \int\theta\sec^2\theta-\int\theta\,d\theta = \theta\tan\theta-\int\tan\theta\,d\theta-\tfrac{1}{2}\theta^2+C$$

$u=\theta$ $\quad v=\tan\theta$

$du=d\theta$ $\quad dv=\sec^2\theta\,d\theta$

$$=\boxed{\theta\tan\theta-\ln|\sec\theta|-\tfrac{1}{2}\theta^2+C}$$

43. $$\int e^x\sqrt{1+e^x}\,dx = \int u^{\frac{1}{2}}\,du = \tfrac{2}{3}u^{\frac{3}{2}}+C = \boxed{\tfrac{2}{3}(1+e^x)^{\frac{3}{2}}+C}$$

$u=1+e^x$

$du=e^x\,dx$

OR 43. $$\int e^x\sqrt{1+e^x}\,dx = \int u\cdot 2u\,du = 2\int u^2\,du = 2\cdot\frac{u^3}{3}+C = \boxed{\tfrac{2}{3}(1+e^x)^{\frac{3}{2}}+C,}$$

$u=\sqrt{1+e^x}$

$u^2=1+e^x$

$2u\,du=e^x\,dx$

45. $$\int x^5e^{-x^3}\,dx = \tfrac{1}{3}\int(-x^3)e^{-x^3}(-3x^2\,dx) = \tfrac{1}{3}\int ye^y\,dy = \tfrac{1}{3}ye^y-\tfrac{1}{3}\int e^y\,dy$$

Parts: $u=y$ $\quad v=e^y$

$du=dy$ $\quad dv=e^y\,dy$

$$=\tfrac{1}{3}ye^y-\tfrac{1}{3}e^y+C$$

$$=\tfrac{1}{3}e^y(y-1)+C$$

$y=-x^3$

$dy=-3x^2\,dx$

$$=\tfrac{1}{3}e^{-x^3}(-x^3-1)=\boxed{-\tfrac{1}{3}e^{-x^3}(x^3+1)+C}$$

47. $\int x^3(x-1)^{-4}\,dx = \int \frac{x^3}{(x-1)^4}\,dx = \int \left[\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}\right] dx$

$x^3 = A(x-1)^3 + B(x-1)^2 + C(x-1) + D$

$x^3 = A(x^3-3x^2+3x-1) + B(x^2-2x+1) + Cx - C + D$

$x^3 = Ax^3 - 3Ax^2 + 3Ax - A + Bx^2 - 2Bx + B + Cx - C + D$

$x^3 = Ax^3 + (-3A+B)x^2 + (3A-2B+C)x + (-A+B-C+D)$

$\boxed{A=1}$

$-3A+B=0$

$-3+B=0$

$\boxed{B=3}$

$3A-2B+C=0$

$3-6+C=0$

$-3+C=0$

$\boxed{C=3}$

$-A+B-C+D=0$

$-1+3-3+D=0$

$\boxed{D=1}$

$= \int \frac{1}{x-1}\,dx + 3\int (x-1)^{-2}\,dx + 3\int (x-1)^{-3}\,dx + \int (x-1)^{-4}\,dx$

$= \ln|x-1| + 3\cdot\frac{(x-1)^{-1}}{-1} + 3\cdot\frac{(x-1)^{-2}}{-2} + \frac{(x-1)^{-3}}{-3} + C$

$= \boxed{\ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} - \frac{1}{3(x-1)^3} + C}$

49. $\frac{1}{4}\int \frac{1\cdot 4}{x\sqrt{4x+1}}\,dx = \frac{1}{\cancel{4}}\int \frac{2\cancel{u}\,du}{\left(\frac{u^2-1}{4}\right)\cdot\cancel{u}} = 2\int \frac{1}{(u+1)(u-1)}\,du = 2\int \left[\frac{A}{u+1} + \frac{B}{u-1}\right] du$

$1 = A(u-1) + B(u+1)$

$1 = (A+B)u + (-A+B)$

$A+B=0$

$-A+B=1$

$2B=1$

$\boxed{B=\frac{1}{2}}$ $\boxed{A=-\frac{1}{2}}$

$u=\sqrt{4x+1}$

$u^2=4x+1$

$2u\,du = 4\,dx$

$\frac{u^2-1}{4} = x$

$= 2\cdot\left(-\frac{1}{2}\right)\ln|u+1| + 2\cdot\frac{1}{2}\ln|u-1| + C$

$= -\ln|u+1| + \ln|u-1| + C$

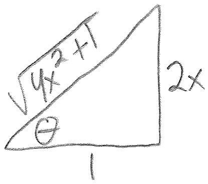
$= \ln\left|\frac{u-1}{u+1}\right| + C = \boxed{\ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C}$

51. $\frac{1}{2}\int \frac{1\cdot 2}{x\sqrt{4x^2+1}}\,dx = \frac{1}{\cancel{2}}\int \frac{\sec^2\theta\,d\theta}{\frac{1}{\cancel{2}}\tan\theta\cdot\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \frac{\left(\frac{1}{\cancel{\cos\theta}}\right)}{\left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)}\,d\theta = \int \csc\theta\,d\theta$

$2x = \tan\theta$

$2\,dx = \sec^2\theta\,d\theta$

$= \ln|\csc\theta - \cot\theta| + C$



$= \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C$

$= \boxed{\ln\left|\frac{\sqrt{4x^2+1}-1}{2x}\right| + C}$

7.5 homework continued

53. $\int x^2 \sinh mx \, dx = \frac{1}{m}x^2 \cosh mx - \int \frac{2}{m} x \cosh mx \, dx$

$u = x^2 \quad v = \frac{\cosh mx}{m}$

$du = 2x\,dx \quad dv = \sinh mx \, dx$

$U = x \quad V = \frac{\sinh mx}{m}$

$dU = dx \quad dV = \cosh mx \, dx$

$= \frac{1}{m}x^2 \cosh mx - \frac{2}{m}\left[\frac{1}{m} x \sinh mx - \int \frac{1}{m} \sinh mx \, dx\right] = \boxed{\frac{1}{m}x^2 \cosh mx - \frac{2}{m^2} x \sinh mx + \frac{2}{m^3} \sinh mx + C}$

55. $\int \frac{dx}{x + x\sqrt{x}} = \int \frac{dx}{x(1+\sqrt{x})} = \int \frac{2\cancel{(u-1)}\,du}{(u-1)^{\cancel{2}} \cdot u} = \int \frac{2}{u(u-1)} du = \int \left[\frac{A}{u} + \frac{B}{u-1}\right] du$

Also would work if I let $u = \sqrt{x}$. $\longrightarrow$ $u = 1 + \sqrt{x} \rightarrow u - 1 = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$2\sqrt{x}\, du = dx$

$2(u-1)\, du = dx$

$2 = A(u-1) + Bu$

$2 = (A+B)u - A$

$\boxed{A = -2}$

$A + B = 0$

$\boxed{B = 2}$

$= -2 \ln|u| + 2 \ln|u-1| + C$

$= -2\ln(1+\sqrt{x}) + 2\ln(1+\sqrt{x}-1) + C$

$= -2\ln(1+\sqrt{x}) + 2\ln(\sqrt{x}) + C$

$= \boxed{2 \ln \left|\frac{\sqrt{x}}{1+\sqrt{x}}\right| + C.}$

57. $\int x \sqrt[3]{x+c}\, dx = \int (u-c) u^{\frac{1}{3}} du = \int \left[u^{\frac{4}{3}} - c u^{\frac{1}{3}}\right] du$

$u = x + c$

$du = dx$

$u - c = x$

$= \frac{3}{7} u^{\frac{7}{3}} - c \cdot \frac{3}{4} u^{\frac{4}{3}} + K = \boxed{\frac{3}{7}(x+c)^{\frac{7}{3}} - \frac{3}{4}c(x+c)^{\frac{4}{3}} + K}$

59. $\int \cos x \cos^3(\sin x)\, dx = \int \cos^3 u \, du = \int (1 - \sin^2 u) \cos u \, du = \int (1 - y^2) dy = y - \frac{1}{3}y^3 + C$

$y = \sin u, \; dy = \cos u \, du$

$u = \sin x, \; du = \cos x \, dx$

$= \sin u - \frac{1}{3}\sin^3 u + C = \boxed{\sin(\sin x) - \frac{1}{3}\sin^3(\sin x) + C,}$

61. $\int \sqrt{x} e^{\sqrt{x}} dx = \int u e^u \cdot 2u\,du = 2\int u^2 e^u du = 2\left[u^2 e^u - \int 2u e^u du\right] = 2u^2 e^u - 4\left[u e^u - \int e^u du\right]$

Parts: $U = u^2 \quad V = e^u$

$dU = 2u\,du \quad dV = e^u du$

$U = u \quad V = e^u$

$dU = du \quad dV = e^u du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$2\sqrt{x}\, du = dx$

$2u\, du = dx$

$= 2u^2 e^u - 4ue^u + 4e^u + C = 2e^u(u^2 - 2u + 2) + C$

$= \boxed{2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C} = \boxed{2e^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C}$

$$63.\ \int \frac{\sin 2x}{1+\cos^4 x}\,dx = (-)\int \frac{\overset{(-)}{2}\sin x\cos x}{1+\cos^4 x}\,dx = \overset{(\frac{1}{2})}{-2}\int \frac{(2)u}{1+u^4}\,du = -\int \frac{\cancel{\sec^2\theta}\,d\theta}{\overset{1}{\cancel{\sec^2\theta}}} = -\theta + C$$

$$= -\tan^{-1}(u^2)+C$$

$$= \boxed{-\tan^{-1}(\cos^2 x)+C}$$

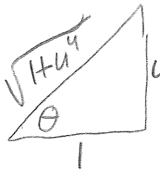
$u = \cos x$

$du = -\sin x\,dx$

$u^2 = \tan\theta$

$2u\,du = \sec^2\theta\,d\theta$

$1+u^4 = 1+\tan^2\theta = \sec^2\theta.$



OR

Shortest →

$$63.\ \int \frac{\sin 2x}{1+\cos^4 x}\,dx = (-)\int \frac{(-)2\sin x\cos x}{1+\cos^4 x}\,dx = -\int \frac{du}{1+u^2} = -\tan^{-1}(u) = \boxed{-\tan^{-1}(\cos^2 x)+C}$$

$u = \cos^2 x$

$du = 2\cos x(-\sin x)\,dx$

OR

$$63.\ \int \frac{\sin 2x}{1+\cos^4 x}\,dx = \int \frac{2\cancel{\sin x}\cancel{\cos x}}{u}\cdot\frac{du}{-4\cos^{\cancel{3}\,2} x\,\cancel{\sin x}} = -\frac{1}{2}\int \frac{1}{u\sqrt{u-1}}\,du = -\frac{1}{\cancel{2}}\int \frac{1}{(y^2+1)\cdot \cancel{y}}\cdot \cancel{2}\cancel{y}\,dy$$

$u = 1+\cos^4 x$

$du = 4\cos^3 x(-\sin x)\,dx$

↑ $\cos^2 x = \sqrt{u-1}$

$y = \sqrt{u-1}$

$y^2 = u-1$

$2y\,dy = du$

$$= -\int \frac{1}{y^2+1}\,dy = -\tan^{-1}(y)+C = -\tan^{-1}(\sqrt{u-1})+C = \boxed{-\tan^{-1}(\cos^2 x)+C}$$

$$65.\ \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx\cdot\frac{[\sqrt{x+1}-\sqrt{x}]}{[\sqrt{x+1}-\sqrt{x}]} = \int \frac{\sqrt{x+1}-\sqrt{x}}{\cancel{x}+1-\cancel{x}}\,dx = \int \sqrt{x+1}-\sqrt{x}\,dx = \boxed{\frac{2}{3}(x+1)^{\frac{3}{2}}-\frac{2}{3}x^{\frac{3}{2}}+C}$$

$$67.\ \int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta}\cdot\sec^2\theta\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta}(1+\tan^2\theta)\,d\theta$$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta}\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta + \Big[\ln|\sec\theta+\tan\theta|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}\,d\theta + \Big[\ln|2+\sqrt{3}|-\ln|\sqrt{2}+1|\Big] = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} u^{-2}\,du + \ln(2+\sqrt{3})-\ln(\sqrt{2}+1)$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \Big[\frac{-1}{u}\Big]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} + \ln(2+\sqrt{3})-\ln(\sqrt{2}+1) = -\frac{2}{\sqrt{3}}+\frac{2}{\sqrt{2}}+\ln(2+\sqrt{3})-\ln(\sqrt{2}+1)$$

$$= \boxed{-\frac{2\sqrt{3}}{3}+\sqrt{2}+\ln(2+\sqrt{3})-\ln(\sqrt{2}+1)}$$

69. $$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{e^x\cdot e^x\,dx}{1+e^x} = \int \frac{(u-1)}{u}\,du = \int\left[1-\frac{1}{u}\right]du = u-\ln|u|+C$$

$$= 1+e^x-\ln|1+e^x|+C = \boxed{e^x-\ln(1+e^x)+C}$$

$u=1+e^x \rightarrow u-1=e^x$

$du=e^x\,dx$

71. $$\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int\frac{-2\cdot x}{\sqrt{1-x^2}}\,dx + \int\frac{\arcsin x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du + \int y\,dy$$

$u=1-x^2$

$du=-2x\,dx$

$y=\arcsin x$

$dy=\frac{1}{\sqrt{1-x^2}}\,dx$

$$= -\frac{1}{\cancel{2}}\cdot\cancel{2}\cdot u^{\frac{1}{2}} + \frac{1}{2}y^2+C$$

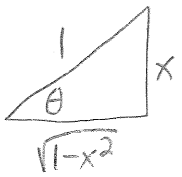
$$= \boxed{-\sqrt{1-x^2}+\frac{1}{2}(\arcsin x)^2+C}$$

OR

71. $$\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx = \int(\sin\theta+\theta)\,d\theta = -\cos\theta+\frac{1}{2}\theta^2+C = \boxed{-\sqrt{1-x^2}+\frac{1}{2}(\arcsin x)^2+C}$$

$\theta=\arcsin x \Rightarrow \sin\theta=x$

$d\theta=\frac{1}{\sqrt{1-x^2}}\,dx$



73. $$\int\frac{1}{(x-2)(x^2+4)}\,dx = \int\left[\frac{A}{x-2}+\frac{Bx+C}{x^2+4}\right]dx$$

$$1 = A(x^2+4)+(Bx+C)(x-2)$$

$$1 = Ax^2+4A+Bx^2-2Bx+Cx-2C$$

$$1 = (A+B)x^2+(-2B+C)x+(4A-2C)$$

$$\begin{array}{l} A+B=0 \\ -2B+C=0 \\ 4A\quad -2C=1 \end{array} \xrightarrow{-4R_1+R_3} \begin{array}{l} A+B=0 \\ -2B+C=0 \\ -4B-2C=1 \end{array} \xrightarrow{-2R_2+R_3} \begin{array}{l} A+B=0 \\ -2B+C=0 \\ -4C=1 \quad \boxed{C=-\frac{1}{4}} \end{array}$$

$$-2B-\frac{1}{4}=0$$

$$-2B=\frac{1}{4}$$

$$\boxed{B=-\frac{1}{8}}$$

$$\boxed{A=\frac{1}{8}}$$

$$= \frac{1}{8}\ln|x-2| + \int\frac{-\frac{1}{8}x-\frac{1}{4}}{x^2+4}\,dx$$

$$= \frac{1}{8}\ln|x-2| - \frac{1}{8}\cdot\frac{1}{2}\int\frac{2\cdot x}{x^2+4}\,dx - \frac{1}{4}\int\frac{1}{x^2+4}\,dx$$

$$= \frac{1}{8}\ln|x-2| - \frac{1}{16}\ln(x^2+4) - \frac{1}{4}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$= \boxed{\frac{1}{8}\ln|x-2| - \frac{1}{16}\ln(x^2+4) - \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right)+C}$$

75. $$\int \frac{xe^x}{\sqrt{1+e^x}}\,dx = \int \frac{\ln(u^2-1)}{\cancel{u}}\cdot 2\cancel{u}du = 2\int \ln(u^2-1)\,du = 2\left[u\ln(u^2-1) - \int \frac{u}{u^2-1}\cdot 2u\,du\right]$$

Parts: $U = \ln(u^2-1)$ $\quad V = u$

$dU = \frac{1}{u^2-1}\cdot 2u\,du$ $\quad dV = du$

$u = \sqrt{1+e^x}$

$u^2 = 1+e^x$

$u^2-1 = e^x \rightarrow \ln(u^2-1) = x$

$2u\,du = e^x\,dx$

$$= 2u\ln(u^2-1) - 4\int \frac{u^2}{u^2-1}\,du = 2u\ln(u^2-1) - 4\int\left[1+\frac{1}{u^2-1}\right]du$$

$$= 2\sqrt{1+e^x}\ln(1+e^x-1) - 4u - 4\int\frac{1\,du}{(u+1)(u-1)}$$

$\nearrow \int\left[\frac{A}{u+1}+\frac{B}{u-1}\right]dx$

$1 = A(u-1)+B(u+1)$

$1 = (A+B)u + (-A+B)$

$A+B=0$

$-A+B=1$

$2B=1$ $\quad \boxed{B=\frac{1}{2}}$

$\boxed{A=-\frac{1}{2}}$

$1+\frac{1}{u^2-1}$

$u^2-1\,\overline{)\,u^2}$

$-(u^2-1)$

1

$$= 2\sqrt{1+e^x}\ln(e^x) - 4\sqrt{1+e^x} - 4\left[-\frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1|\right] + C$$

$$= 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} + 2\ln|\sqrt{1+e^x}+1| - 2\ln|\sqrt{1+e^x}-1| + C$$

77. $$\frac{2}{3}\int\frac{\frac{3}{2}\sqrt{x}}{1+x^3}\,dx = \frac{2}{3}\int\frac{du}{1+u^2} = \frac{2}{3}\tan^{-1}u + C = \boxed{\frac{2}{3}\tan^{-1}(x^{\frac{3}{2}})+C}$$

tricky!

creative substitution to set it up for arctan.

$u = x^{\frac{3}{2}}$

$u^2 = x^3$

$du = \frac{3}{2}x^{\frac{1}{2}}dx$

$du = \frac{3}{2}\sqrt{x}\,dx$

79. $$\int x\sin^2 x\cos x\,dx = \frac{1}{3}x\sin^3 x - \int\frac{1}{3}\sin^3 x\,dx = \frac{1}{3}x\sin^3 x \overset{(-)}{-}\frac{1}{3}\int(1-\cos^2 x)^{(-)}\sin x\,dx$$

$z = \cos x$

$dz = -\sin x\,dx$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int(1-z^2)\,dz$$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\left[z - \frac{1}{3}z^3\right] + C$$

$$= \boxed{\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C}$$

Parts: $u = x$ $\quad v = \frac{1}{3}\sin^3 x$

$du = dx$ $\quad dv = \sin^2 x\cos x\,dx$

$\hookrightarrow v = \int\sin^2 x\cos x\,dx = \int y^2\,dy = \frac{1}{3}y^3$

$y = \sin x,\ dy = \cos x\,dx$