Expression	substitution	identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

1. $$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{\cancel{3}\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\cdot\cancel{3}\tan\theta} = \frac{1}{9}\int\frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int\cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$$

$x = 3\sec\theta$

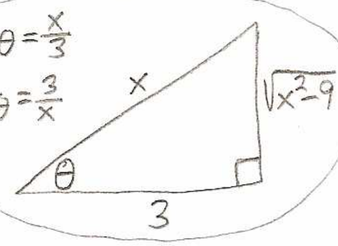
$dx = 3\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-9} = \sqrt{9\sec^2\theta-9} = \sqrt{9(\sec^2\theta-1)}$

$= \sqrt{9\tan^2\theta} = 3\tan\theta$

$\sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x}$



$$= \boxed{\frac{1}{9}\cdot\frac{\sqrt{x^2-9}}{x} + C}$$

3. $$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{27\tan^3\theta}{\cancel{3}\sec\theta}\cdot\cancel{3}\sec^2\theta\,d\theta = 27\int\tan^3\theta\sec\theta\,d\theta = 27\int\tan^2\theta\cdot\sec\theta\tan\theta\,d\theta$$

$$= 27\int(\sec^2\theta-1)\cdot\sec\theta\tan\theta\,d\theta = 27\int(u^2-1)\,du = 27\left[\frac{1}{3}u^3-u\right]+C$$

$u=\sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$= 9\sec^3\theta - 27\sec\theta + C$$

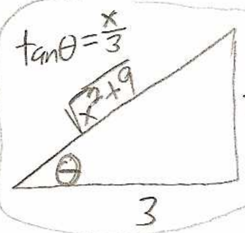
$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9(\tan^2\theta+1)}$

$= \sqrt{9\sec^2\theta} = 3\sec\theta$

$\tan\theta = \frac{x}{3}$



$$= 9\cdot\frac{(x^2+9)^{\frac{3}{2}}}{27} - 27\frac{\sqrt{x^2+9}}{3} + C$$

$$= \boxed{\frac{1}{3}(x^2+9)^{\frac{3}{2}} - 9\sqrt{x^2+9} + C}$$

$$= \frac{1}{3}\sqrt{x^2+9}\left[(x^2+9)-27\right]+C = \boxed{\frac{1}{3}\sqrt{x^2+9}\,(x^2-18)+C}$$

<u>OR</u> (regular u-substitution)

3. $$\frac{1}{2}\int \frac{x^3\,(2)}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int\frac{u-9}{\sqrt{u}}\,du = \frac{1}{2}\int\left(u^{\frac{1}{2}}-9u^{-\frac{1}{2}}\right)du = \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} - 9\cdot 2u^{\frac{1}{2}}\right]+C$$

$u = x^2+9$

$du = 2x\,dx$

$u-9 = x^2$

$$= \boxed{\frac{1}{3}(x^2+9)^{\frac{3}{2}} - 9\sqrt{x^2+9} + C} = \frac{1}{3}\sqrt{x^2+9}\left[(x^2+9)-27\right]+C$$

$$= \boxed{\frac{1}{3}\sqrt{x^2+9}\left[x^2-18\right]+C}$$

5. $$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta \cancel{\tan\theta}\, d\theta}{\sec^3\theta \cdot \cancel{\tan\theta}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$t = \sec\theta$

$dt = \sec\theta\tan\theta\, d\theta$

$\sqrt{t^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta}$
$= \tan\theta$

t	θ
$\sqrt{2}$	$\frac{\pi}{4}$
2	$\frac{\pi}{3}$

$\sec\theta = t$

$\cos\theta = \frac{1}{t}$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{2}\left[\left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{4} + \frac{1}{2}\cdot 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{4} - \frac{\pi}{4} - \frac{1}{2}\right] = \frac{1}{2}\left[\frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2}\right] = \boxed{\frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}}$$

7. $$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{\cancel{5}\cos\theta\, d\theta}{25\sin^2\theta \cdot \cancel{5\cos\theta}} = \frac{1}{25}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{25}\int \csc^2\theta\, d\theta = -\frac{1}{25}\cot\theta + C$$

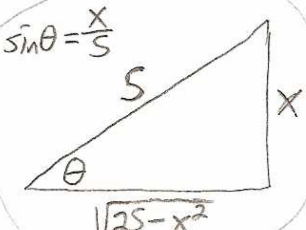
$\frac{d}{dx}\cot\theta = -\csc^2\theta$

$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$\sqrt{25-x^2} = \sqrt{25 - 25\sin^2\theta} = \sqrt{25(1-\sin^2\theta)}$
$= \sqrt{25\cos^2\theta} = 5\cos\theta$

$\sin\theta = \frac{x}{5}$



$$= \boxed{-\frac{1}{25}\cdot\frac{\sqrt{25-x^2}}{x} + C}$$

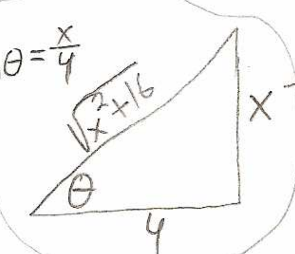
9. $$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{\cancel{4}\sec^2\theta\, d\theta}{\cancel{4}\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$\sqrt{x^2+16} = \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta + 1)}$
$= \sqrt{16\sec^2\theta} = 4\sec\theta.$

$\tan\theta = \frac{x}{4}$



$$= \boxed{\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C}$$

$$= \ln\left|\frac{\sqrt{x^2+16} + x}{4}\right| + C$$

$$= \ln\left|\sqrt{x^2+16} + x\right| \underbrace{- \ln 4 + C}_{\text{a constant}}$$

$$= \boxed{\ln\left|\sqrt{x^2+16} + x\right| + C}$$

11. $\int \sqrt{1-4x^2}\,dx = \left(\frac{1}{2}\right)\int \sqrt{1-(2x)^2}\,dx\,(2) = \frac{1}{2}\int \sqrt{1-u^2}\,du = \frac{1}{2}\int \cos\theta \cdot \cos\theta\,d\theta = \frac{1}{2}\int \cos^2\theta\,d\theta$

$u = 2x$
$du = 2dx$

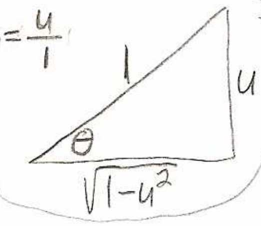
$u = \sin\theta$
$du = \cos\theta\,d\theta$
$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta.$

$= \frac{1}{2}\int \frac{1}{2}[1+\cos 2\theta]\,d\theta$

$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$

$= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{4}\theta + \frac{1}{8}\cdot 2\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}u + \frac{1}{4}\cdot u\cdot\sqrt{1-u^2} + C$

$\sin\theta = u = \frac{u}{1}$



$= \frac{1}{4}\sin^{-1}(2x) + \frac{1}{4}\cdot 2x\cdot\sqrt{1-4x^2} + C$

$= \boxed{\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C}$

OR

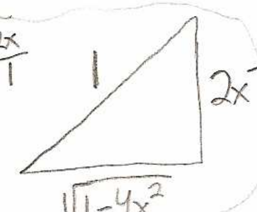
11. $\int \sqrt{1-4x^2}\,dx = \left(\frac{1}{2}\right)\int \sqrt{1-(2x)^2}\,dx^{(2)} = \frac{1}{2}\int \cos\theta\cdot\cos\theta\,d\theta = \frac{1}{2}\int \cos^2\theta\,d\theta = \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$

$2x = \sin\theta$
$2dx = \cos\theta\,d\theta$
$\sqrt{1-4x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right] + C = \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C$

$= \frac{1}{4}\theta + \frac{1}{8}\cdot 2\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}(2x) + \frac{1}{4}\cdot 2x\cdot\sqrt{1-4x^2} + C$

$\sin\theta = 2x = \frac{2x}{1}$



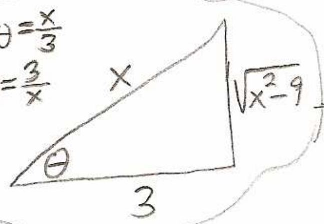
$= \boxed{\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C}$

13. $\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\cdot 3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{3}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\,d\theta = \frac{1}{3}\int \sin^2\theta\,d\theta$

$x = 3\sec\theta$
$dx = 3\sec\theta\tan\theta\,d\theta$
$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9}$
$= \sqrt{9(\sec^2\theta - 1)}$
$= \sqrt{9\tan^2\theta}$
$= 3\tan\theta$

$= \frac{1}{3}\int \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{6}\left[\theta - \frac{\sin 2\theta}{2}\right] + C = \frac{1}{6}\theta - \frac{1}{12}\cdot 2\sin\theta\cos\theta + C$

$\sec\theta = \frac{x}{3}$
$\cos\theta = \frac{3}{x}$



$= \frac{1}{6}\cos^{-1}\left(\frac{3}{x}\right) - \frac{1}{6}\cdot\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x} + C$

$= \boxed{\frac{1}{6}\cos^{-1}\left(\frac{3}{x}\right) - \frac{1}{2}\cdot\frac{\sqrt{x^2-9}}{x^2} + C}$

15. $\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} a^2\sin^2\theta\cdot a\cos\theta\cdot a\cos\theta\,d\theta = a^4\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta$

$x = a\sin\theta$
$dx = a\cos\theta\,d\theta$
$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$
$= \sqrt{a^2(1-\sin^2\theta)}$
$= \sqrt{a^2\cos^2\theta}$
$= a\cos\theta.$

x	θ
0	0
a	$\frac{\pi}{2}$

$x = a\sin\theta$, so:
$0 = a\sin\theta \Rightarrow 0 = \sin\theta \Rightarrow \theta = 0.$
$a = a\sin\theta \Rightarrow 1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}.$

$= a^4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)\cdot\frac{1}{2}(1+\cos 2\theta)\,d\theta$

$= \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\left[1-\cos^2 2\theta\right]d\theta$

$= \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\left[1-\frac{1}{2}(1+\cos 4\theta)\right]d\theta = \frac{1}{4}a^4\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}-\frac{1}{2}\cos 4\theta\right]d\theta$

$= \frac{1}{4}a^4\left[\frac{1}{2}\theta - \frac{1}{2}\cdot\frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{1}{4}a^4\left[\left(\frac{\pi}{4}-0\right)-(0-0)\right] = \boxed{\frac{a^4\pi}{16}}$

17. $\left(\frac{1}{2}\right)\int \frac{x}{\sqrt{x^2-7}}\,2\,dx = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\cdot 2u^{\frac{1}{2}} + C = \boxed{\sqrt{x^2-7}+C}$

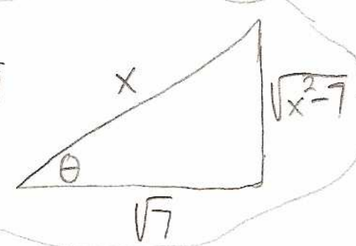
Basic u-substitution: $u = x^2-7$
$du = 2x\,dx$

OR

17. $\int \frac{x}{\sqrt{x^2-7}}\,dx = \int \frac{\sqrt{7}\sec\theta}{\sqrt{7}\tan\theta}\cdot\sqrt{7}\sec\theta\cancel{\tan\theta}\,d\theta = \sqrt{7}\int\sec^2\theta\,d\theta = \sqrt{7}\tan\theta + C$

$x = \sqrt{7}\sec\theta$
$dx = \sqrt{7}\sec\theta\tan\theta\,d\theta$
$\sqrt{x^2-7} = \sqrt{7\sec^2\theta-7}$
$= \sqrt{7(\sec^2\theta-1)}$
$= \sqrt{7\tan^2\theta}$
$= \sqrt{7}\tan\theta.$

$\sec\theta = \frac{x}{\sqrt{7}}$
$\cos\theta = \frac{\sqrt{7}}{x}$



$= \sqrt{7}\cdot\frac{\sqrt{x^2-7}}{\sqrt{7}} + C$

$= \boxed{\sqrt{x^2-7}+C}$

19. $\int\frac{\sqrt{1+x^2}}{x}\,dx = \int\frac{\sec\theta}{\tan\theta}\cdot\sec^2\theta\,d\theta = \int\frac{\sec\theta}{\tan\theta}\cdot(1+\tan^2\theta)\,d\theta = \int\left[\frac{\sec\theta}{\tan\theta}+\sec\theta\tan\theta\right]d\theta$

$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$
$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta}$
$= \sec\theta.$

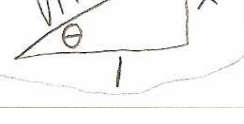
$= \int\frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{\sin\theta}\,d\theta + \int\sec\theta\tan\theta\,d\theta = \int\csc\theta\,d\theta + \int\sec\theta\tan\theta\,d\theta$

$= \ln|\csc\theta-\cot\theta| + \sec\theta + C = \boxed{\ln\left|\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right| + \sqrt{1+x^2}+C}$

$= \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2}+C$

$= \boxed{\ln\left|\sqrt{1+x^2}-1\right| - \ln|x| + \sqrt{1+x^2}+C}$

$\tan\theta = x = \frac{x}{1}$



21. $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{0.6} \frac{x^2}{\sqrt{3^2-(5x)^2}}\,dx = \int_0^{\frac{\pi}{2}} \frac{\frac{9}{25}\sin^2\theta}{\cancel{3}\cos\theta}\cdot\frac{\cancel{3}}{5}\cos\theta\,d\theta$

$5x = 3\sin\theta \rightarrow x = \frac{3}{5}\sin\theta$

$5dx = 3\cos\theta\,d\theta$

$dx = \frac{3}{5}\cos\theta\,d\theta$

$\sqrt{9-25x^2} = \sqrt{9-9\sin^2\theta}$

$= \sqrt{9(1-\sin^2\theta)}$

$= 3\cos\theta.$

x	θ	
0	0	$\leftarrow 0 = \frac{3}{5}\sin\theta \Rightarrow 0 = \sin\theta \Rightarrow \theta = 0.$
0.6	$\frac{\pi}{2}$	$\leftarrow 0.6 = \frac{3}{5}\sin\theta \Rightarrow 1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}.$

$$= \frac{9}{125}\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \frac{9}{125}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{9}{250}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{9}{250}\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right]$$

$$= \boxed{\frac{9\pi}{500}}$$

23. $\displaystyle\int \sqrt{5+4x-x^2}\,dx = \int \sqrt{9-(x-2)^2}\,dx = \int 3\cos\theta\cdot 3\cos\theta\,d\theta = 9\int \cos^2\theta\,d\theta$

$5+4x-x^2 = -x^2+4x+5$

$= -(x^2-4x \qquad)+5$

$= -(x^2-4x+4-4)+5$

$= -(x^2-4x+4)+4+5$

$= -(x-2)^2+9$

$x-2 = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$\sqrt{9-(x-2)^2} = \sqrt{9-9\sin^2\theta}$

$= \sqrt{9(1-\sin^2\theta)}$

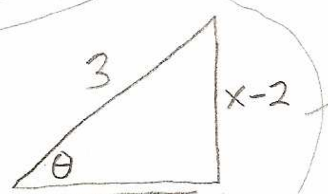
$= 3\cos\theta.$

$= 9\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$

$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$

$= \frac{9}{2}\left[\theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$

$\sin\theta = \frac{x-2}{3}$



$\sqrt{9-(x-2)^2}$

$= \sqrt{5+4x-x^2}$

$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3}\right]$

$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C}$

25. $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{x}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}\,dx = \int \frac{\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}}{\cancel{\frac{\sqrt{3}}{2}}\sec\theta}\cdot \cancel{\frac{\sqrt{3}}{2}}\sec^{\cancel{2}}\theta\,d\theta = \int \left(\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}\right)\sec\theta\,d\theta$

$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1$
$= (x+\frac{1}{2})^2+\frac{3}{4}$

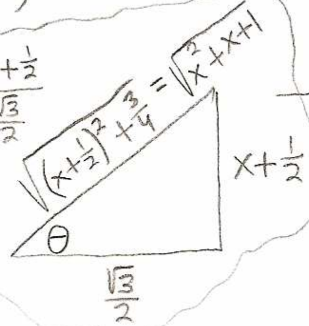
OR

$\frac{1}{4}\left[(4x^2+4x+1)+3\right]$
$= \frac{1}{4}\left[(2x+1)^2+3\right]$

$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$
$dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$
$\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}} = \sqrt{\frac{3}{4}\tan^2\theta+\frac{3}{4}} = \sqrt{\frac{3}{4}(\tan^2\theta+1)}$
$= \frac{\sqrt{3}}{2}\sec\theta.$

$= \frac{\sqrt{3}}{2}\int \sec\theta\tan\theta\,d\theta - \frac{1}{2}\int\sec\theta\,d\theta = \frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$

$\tan\theta = \dfrac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$



$= \cancel{\frac{\sqrt{3}}{2}}\cdot \dfrac{\sqrt{x^2+x+1}}{\cancel{\frac{\sqrt{3}}{2}}} - \frac{1}{2}\ln\left|\dfrac{\sqrt{x^2+x+1}}{\frac{\sqrt{3}}{2}} + \dfrac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right| + C$

$= \boxed{\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\dfrac{\sqrt{x^2+x+1}+x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right| + C}$

$= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1}+x+\frac{1}{2}\right| \underbrace{-\frac{1}{2}\ln\frac{\sqrt{3}}{2} + C}_{\text{constant}} = \boxed{\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1}+x+\frac{1}{2}\right| + C}$

27. $\displaystyle\int\sqrt{x^2+2x}\,dx = \int\sqrt{(x+1)^2-1}\,dx = \int \tan\theta\cdot\sec\theta\tan\theta\,d\theta = \int\tan^2\theta\sec\theta\,d\theta = \int(\sec^2\theta-1)\sec\theta\,d\theta$

$x^2+2x = x^2+2x+1-1$
$= (x+1)^2-1$

$x+1 = \sec\theta$
$dx = \sec\theta\tan\theta\,d\theta$
$\sqrt{(x+1)^2-1} = \sqrt{\sec^2\theta-1} = \tan\theta.$

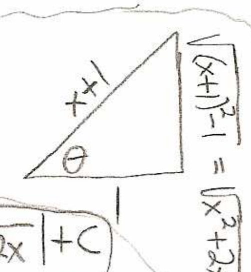
$= \int\sec^3\theta\,d\theta - \int\sec\theta\,d\theta$

$\overset{*}{=} \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right] - \ln|\sec\theta+\tan\theta| + C$

$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta| + C$

$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$

$\sec\theta = x+1$
$\cos\theta = \frac{1}{x+1}$



$= \boxed{\frac{1}{2}\cdot(x+1)\sqrt{x^2+2x} - \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x}\right| + C}$

$\displaystyle * \int\sec^3\theta\,d\theta = \int\sec\theta\cdot\sec^2\theta\,d\theta$

$u = \sec\theta \quad v = \tan\theta$
$du = \sec\theta\tan\theta\,d\theta \quad dv = \sec^2\theta\,d\theta$

$\int\sec^3\theta\,d\theta = \sec\theta\tan\theta - \int\tan^2\theta\sec\theta\,d\theta$

$\int\sec^3\theta\,d\theta = \sec\theta\tan\theta - \int(\sec^2\theta-1)\sec\theta\,d\theta$

$\int\sec^3\theta\,d\theta = \sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta$

$2\int\sec^3\theta\,d\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|$

$\int\sec^3\theta\,d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right] + C$

29. $\int x\sqrt{1-x^4}\,dx \overset{\left(\frac{1}{2}\right)}{=} \int x\sqrt{1-(x^2)^2}\,dx^{(2)} = \frac{1}{2}\int \cos\theta\cdot\cos\theta\,d\theta = \frac{1}{2}\int \cos^2\theta\,d\theta = \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$

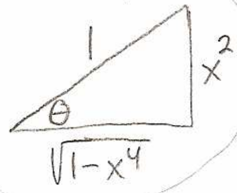
$x^2 = \sin\theta$

$2x\,dx = \cos\theta\,d\theta$

$\sqrt{1-x^4} = \sqrt{1-\sin^2\theta} = \cos\theta.$

$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right] + C = \frac{1}{4}\theta + \frac{1}{8}\cdot 2\sin\theta\cos\theta + C$

$\sin\theta = x^2 = \frac{x^2}{1}$



$= \boxed{\frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}\cdot x^2\cdot\sqrt{1-x^4} + C}$

33. $f(x) = \frac{\sqrt{x^2-1}}{x}$, $1 \le x \le 7$. $f_{avg} = \frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int \frac{\tan\theta}{\cancel{\sec\theta}}\cdot\cancel{\sec\theta}\tan\theta\,d\theta = \frac{1}{6}\int \tan^2\theta\,d\theta$

$= \frac{1}{6}\int\left[\sec^2\theta - 1\right]d\theta$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

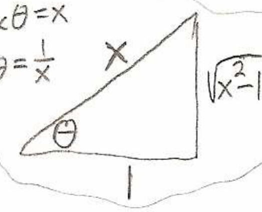
$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta.$

changing limits is too weird here. I will convert back to x and use the original limits at the end.

$= \frac{1}{6}\left[\tan\theta - \theta\right]$

$\sec\theta = x$

$\cos\theta = \frac{1}{x}$



$= \frac{1}{6}\left[\sqrt{x^2-1} - \cos^{-1}\left(\frac{1}{x}\right)\right]_1^7 = \frac{1}{6}\left[\left(\sqrt{48} - \cos^{-1}\left(\frac{1}{7}\right)\right) - \left(0 - \cos^{-1}1\right)\right]$

$= \frac{1}{6}\left[4\sqrt{3} - \cos^{-1}\left(\frac{1}{7}\right) + 0\right]$

$= \boxed{\frac{2\sqrt{3}}{3} - \frac{1}{6}\cos^{-1}\left(\frac{1}{7}\right)}$