7.2 homework

1. $\int \sin^3 x \cos^2 x\,dx = \int \sin x(1-\cos^2 x)\cos^2 x\,dx = -\int(\cos^2 x - \cos^4 x)\overset{(-1)}{\sin x}\,dx = -\int u^2 - u^4\,du = -\left[\frac{u^3}{3} - \frac{u^5}{5}\right] = \boxed{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.}$

$u = \cos x \quad du = -\sin x\,dx$

3. $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin^5 x \cos^3 x\,dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin^5 x(1-\sin^2 x)\cos x\,dx = \int_1^{\frac{\sqrt{2}}{2}} u^5 - u^7\,du = \left[\frac{u^6}{6} - \frac{u^8}{8}\right]_1^{\frac{\sqrt{2}}{2}} = \left(\frac{1}{6}\cdot\frac{1}{8} - \frac{1}{8}\cdot\frac{1}{16}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) = \frac{1}{48} - \frac{1}{128} - \frac{2}{48}$

$u = \sin x \quad du = \cos x\,dx$

$48 = 2\cdot 3\cdot 2^3 = 2^4\cdot 3$

$128 = 2^7 \quad LCD = 2^7\cdot 3$

$= -\frac{1}{48} - \frac{1}{128} = \frac{-8-3}{384}$

$= \boxed{\frac{-11}{384}}.$

5. $\int \sin^2(\pi x)\cos^5(\pi x)\,dx = \frac{1}{\pi}\int \sin^2 \pi x(1-\sin^2 \pi x)^2 \cos(\pi x)\overset{(\pi)}{dx} = \frac{1}{\pi}\int u^2(1-u^2)^2\,du = \frac{1}{\pi}\int u^2(1-2u^2+u^4)\,du$

$u = \sin \pi x \quad du = \pi\cos \pi x$

$= \frac{1}{\pi}\int(u^2 - 2u^4 + u^6)\,du = \frac{1}{\pi}\left[\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right] + C = \boxed{\frac{1}{\pi}\left[\frac{1}{3}\sin^3 \pi x - \frac{2}{5}\sin^5 \pi x + \frac{1}{7}\sin^7 \pi x\right] + C}$

7. $\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \boxed{\frac{\pi}{4}}$

9. $\int_0^{\pi} \sin^4(3t)\,dt = \int_0^{\pi} \sin^2 3t\cdot\sin^2 3t\,dt = \int_0^{\pi} \frac{1}{2}[1-\cos 6t]\cdot\frac{1}{2}[1-\cos 6t]\,dt = \int_0^{\pi}\frac{1}{4}[1 - 2\cos 6t + \cos^2 6t]\,dt$

$= \int_0^{\pi}\left(\frac{1}{4} - \frac{1}{2}\cos 6t + \frac{1}{4}\cdot\frac{1}{2}[1+\cos 12t]\right)dt = \int_0^{\pi}\left(\frac{1}{4} - \frac{1}{2}\cos 6t + \frac{1}{8} + \frac{1}{8}\cos 12t\right)dt = \left[\frac{3}{8}t - \frac{1}{12}\sin 6t - \frac{1}{96}\sin 12t\right]_0^{\pi}$

$= \left(\frac{3}{8}\pi - 0 - 0\right) - (0-0-0) = \boxed{\frac{3\pi}{8}.}$

11. $\int(1+\cos\theta)^2\,d\theta = \int\left(1 + 2\cos\theta + \frac{1}{2}[1+\cos 2\theta]\right)d\theta = \int\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \boxed{\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C}$

13. $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}[1-\cos 2x]\cdot\frac{1}{2}[1+\cos 2x]\,dx = \int_0^{\frac{\pi}{2}}\frac{1}{4}[1-\cos^2 2x]\,dx = \int_0^{\frac{\pi}{2}}\frac{1}{4}\left[1 - \frac{1}{2}(1+\cos 4x)\right]dx$

$= \int_0^{\frac{\pi}{2}}\left[\frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 4x\right]dx = \left[\frac{1}{8}x - \frac{1}{32}\sin 4x\right]_0^{\frac{\pi}{2}} = \left(\frac{\pi}{16} - 0\right) - (0-0) = \boxed{\frac{\pi}{16}}$

15. $\int \frac{\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha = \int\frac{(1-\sin^2\alpha)^2\cos\alpha}{\sqrt{\sin\alpha}}\,d\alpha = \int\frac{(1-u^2)^2}{\sqrt{u}}\,du = \int\frac{1-2u^2+u^4}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}}\,du$

$u = \sin\alpha \quad du = \cos\alpha\,d\alpha$

$= 2u^{\frac{1}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{9}u^{\frac{9}{2}} + C \rightarrow = \frac{2}{45}\sqrt{u}\left[45 - 18u^2 + 5u^4\right] + C = \boxed{\frac{2}{45}\sqrt{\sin\alpha}\left[45 - 18\sin^2\alpha + 5\sin^4\alpha\right] + C}$

OR $= \boxed{2\sqrt{\sin\alpha} - \frac{4}{5}(\sin\alpha)^{\frac{5}{2}} + \frac{2}{9}(\sin\alpha)^{\frac{9}{2}} + C}$

17. $\int\cos^2 x\tan^3 x\,dx = \int\cos^2 x\cdot\frac{\sin^3 x}{\cos^3 x}\,dx = \int\frac{\sin^3 x}{\cos x}\,dx = -\int\frac{1-\cos^2 x}{\cos x}\cdot\overset{(-1)}{\sin x}\,dx = -\int\frac{1-u^2}{u}\,du = -\int\left(\frac{1}{u} - u\right)du$

$u = \cos x \quad du = -\sin x$

$= -\ln|u| + \frac{u^2}{2} + C = \boxed{-\ln|\cos x| + \frac{1}{2}\cos^2 x + C}$

19. $\int\frac{\cos x + \sin 2x}{\sin x}\,dx = \int\frac{\cos x + 2\sin x\cos x}{\sin x}\,dx = \int\frac{\cos x}{\sin x} + 2\cos x\,dx = \ln|\sin x| + 2\sin x + C$

$u = \sin x \quad du = \cos x\,dx$

OR: $\int\frac{(1+2\sin x)}{\sin x}\cos x\,dx = \int\frac{1+2u}{u}\,du = \int\left(\frac{1}{u} + 2\right)du = \ln|u| + 2u + C = \boxed{\ln|\sin x| + 2\sin x + C}$

$u = \sin x$

$du = \cos x\,dx$

21. $\int \sec^2 x \tan x\,dx = \int \sec x \cdot \sec x \tan x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sec^2 x + C}$

$u = \sec x,\ du = \sec x \tan x\,dx$

OR:

$\int \sec^2 x \tan x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2 x + C}$

$u = \tan x,\ du = \sec^2 x\,dx$

23. $\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \boxed{\tan x - x + C}$

25. $\int \sec^6 t\,dt = \int (1+\tan^2 t)^2 \sec^2 t\,dt = \int (1+u^2)^2 du = \int 1 + 2u^2 + u^4\,du = u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$

$u = \tan t,\ du = \sec^2 t\,dt$

$= \boxed{\tan t + \frac{2}{3}\tan^3 t + \frac{1}{5}\tan^5 t + C}$

27. $\int_0^{\frac{\pi}{3}} \tan^5 x \sec^4 x\,dx = \int_0^{\frac{\pi}{3}} \tan^5 x (1+\tan^2 x)\sec^2 x\,dx = \int_0^{\sqrt{3}} (u^5 + u^7)\,du = \left[\frac{u^6}{6} + \frac{u^8}{8}\right]_0^{\sqrt{3}} = \frac{27}{6} + \frac{81}{8} = \frac{108+243}{24}$

$u = \tan x,\ du = \sec^2 x\,dx$

$= \frac{351}{24} = \boxed{\frac{117}{8}}$

29. $\int \tan^3 x \sec x\,dx = \int (\sec^2 x - 1)\sec x \tan x\,dx = \int u^2 - 1\,du = \frac{u^3}{3} - u + C = \boxed{\frac{1}{3}\sec^3 x - \sec x + C}$

$u = \sec x,\ du = \sec x \tan x\,dx$

31. $\int \tan^5 x\,dx = \int \tan^3 x \tan^2 x\,dx = \int \tan^3 x(\sec^2 x - 1)\,dx = \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx = \int u^3 du - \int \tan x(\sec^2 x - 1)\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$= \frac{u^4}{4} - \int \tan x \sec^2 x\,dx + \int \tan x\,dx = \boxed{\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C}$ OR $\boxed{\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C}$

Check: $\tan^3 x \cdot \sec^2 x - \tan x \sec^2 x - \frac{1}{\cos x}\cdot(-\sin x) = \tan^3 x \sec^2 x - \tan x \sec^2 x + \tan x = \tan x(\tan^2 x \sec^2 x - \sec^2 x + 1)$

$= \frac{\sin x}{\cos x}\left[\frac{\sin^2 x}{\cos^4 x} - \frac{1}{\cos^2 x} + \frac{\cos^4 x}{\cos^4 x}\right] = \frac{\sin x}{\cos x}\left[\frac{\sin^2 x}{\cos^4 x} - \frac{\cos^2 x}{\cos^4 x} + \frac{(1-\sin^2 x)^2}{\cos^4 x}\right] = \frac{\sin x}{\cos x}\left[\frac{\sin^2 x - \cos^2 x + 1 - 2\sin^2 x + \sin^4 x}{\cos^4 x}\right]$

$= \frac{\sin x}{\cos x}\left[\frac{\cancel{\sin^2 x} - (\cancel{1} - \cancel{\sin^2 x}) + \cancel{1} - \cancel{2\sin^2 x} + \sin^4 x}{\cos^4 x}\right] = \tan^5 x.$ ✓

OR

Check: $\tan^3 x \sec^2 x - \tan x \sec^2 x + \tan x = \tan^3 x(1+\tan^2 x) - \tan x(1+\tan^2 x) + \tan x = \cancel{\tan^3 x} + \tan^5 x - \cancel{\tan x} - \cancel{\tan^3 x} + \cancel{\tan x}.$ ✓

Another way → 31. $\int \tan^5 x\,dx = \int \tan^2 x \tan^2 x \tan x\,dx = \int (\sec^2 x - 1)^2 \tan x\,dx = \int [\sec^4 x - 2\sec^2 x + 1]\tan x\,dx$

$= \int \sec^4 x \tan x\,dx - 2\int \tan x \sec^2 x\,dx + \int \tan x\,dx = \int \sec^3 x \cdot \sec x \tan x - 2\cdot\frac{\tan^2 x}{2} + \ln|\sec x|$

$= \boxed{\frac{\sec^4 x}{4} - \tan^2 x + \ln|\sec x| + C}$

This term could be $-\ln|\cos x|$ as well.

this term could be $-\sec^2 x$ as well.

7.2 homework continued

33. $\int \frac{\tan^3\theta}{\cos^4\theta}\,d\theta = \int \tan^3\theta \sec^4\theta\,d\theta = \int \tan^3\theta(1+\tan^2\theta)\sec^2\theta\,d\theta = \int u^3+u^5\,du = \frac{u^4}{4}+\frac{u^6}{6}+C = \boxed{\frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta + C}$

$u=\tan\theta,\ du=\sec^2\theta\,d\theta$

35. $\int x\sec x\tan x\,dx = (\text{Parts})\ x\sec x - \int \sec x\,dx = \boxed{x\sec x - \ln|\sec x+\tan x| + C}$

$u=x \quad v=\sec x$

$du=dx \quad dv=\sec x\tan x\,dx$

37. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2 x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}[\csc^2 x-1]\,dx = \left[-\cot x - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left(-\cot\frac{\pi}{2}-\frac{\pi}{2}\right)-\left(-\cot\frac{\pi}{6}-\frac{\pi}{6}\right) = \left(0-\frac{\pi}{2}\right)+\sqrt{3}+\frac{\pi}{6} = \sqrt{3}+\frac{\pi}{6}-\frac{\pi}{2}$

$= \boxed{\sqrt{3}-\frac{\pi}{3}}$

$1+\cot^2 x = \csc^2 x$

39. $\int \cot^3\alpha\csc^3\alpha\,d\alpha = \int \cot\alpha\cot^2\alpha\csc^3\alpha\,d\alpha = \int \cot\alpha(\csc^2\alpha-1)\csc^3\alpha\,d\alpha = \int \cot\alpha[\csc^5\alpha-\csc^3\alpha]\,d\alpha$

$= \int \csc^5\alpha\cot\alpha\,d\alpha - \int \csc^3\alpha\cot\alpha\,d\alpha = -\int \csc^4\alpha(-\csc\alpha\cot\alpha\,d\alpha) + \int \csc^2\alpha(-\csc\alpha\cot\alpha\,d\alpha)$

$\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\csc x = -\csc x\cot x$

$= -\int u^4\,du + \int u^2\,du = -\frac{u^5}{5}+\frac{u^3}{3} = \boxed{-\frac{1}{5}\csc^5\alpha+\frac{1}{3}\csc^3\alpha+C}$

OR Faster: $\int \cot^3\alpha\csc^3\alpha\,d\alpha = \int \cot\alpha\cot^2\alpha\csc^2\alpha\,d\alpha = \int \underbrace{\cot\alpha(\csc^2\alpha-1)\csc^2\alpha\csc\alpha\,d\alpha} = -\int[\csc^4\alpha-\csc^2\alpha][-\csc\alpha\cot\alpha\,d\alpha]$

$= -\int(u^4-u^2)\,du = -\frac{1}{5}u^5+\frac{1}{3}u^3+C = \boxed{-\frac{1}{5}\csc^5\alpha+\frac{1}{3}\csc^3\alpha+C}$

41. $\int \csc x\,dx = \int \csc x\cdot\frac{\csc x-\cot x}{\csc x-\cot x}\,dx = \int \frac{\csc^2 x-\csc x\cot x}{\csc x-\cot x}\,dx$

$u=\csc x-\cot x$

$du = -\csc x\cot x-(-\csc^2 x)\,dx$

$= \int \frac{du}{u} = \ln|u| = \boxed{\ln|\csc x-\cot x|+C}$

43. $\int \sin 8x\cos 5x\,dx = \frac{1}{2}\int[\sin 3x+\sin 13x]\,dx = \frac{1}{2}\left[\frac{-\cos 3x}{3}\right]+\frac{1}{2}\left[\frac{-\cos 13x}{13}\right]+C = \boxed{-\frac{1}{6}\cos 3x-\frac{1}{26}\cos 13x+C}$

$\sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$ product-sum id.

45. $\int \sin 5\theta\sin\theta\,d\theta = \frac{1}{2}\int[\cos 4\theta-\cos 6\theta]\,d\theta = \frac{1}{2}\left[\frac{\sin 4\theta}{4}-\frac{\sin 6\theta}{6}\right]+C = \boxed{\frac{1}{8}\sin 4\theta-\frac{1}{12}\sin 6\theta+C}$

$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$ product-sum id.

47. $\int \frac{1-\tan^2 x}{\sec^2 x}\,dx = \int \cos^2 x-\sin^2 x\,dx = \int \cos 2x\,dx = \boxed{\frac{1}{2}\sin 2x+C}$

49. $\frac{1}{2}\int t\sec^2(t^2)\tan^4(t^2)\,dt(2) = \frac{1}{2}\int \sec^2 u\tan^4 u\,du = \frac{1}{2}\int a^4\,da = \frac{1}{2}\cdot\frac{a^5}{5}+C = \frac{1}{10}\tan^5 u+C = \boxed{\frac{1}{10}\tan^5(t^2)+C}$

$u=t^2$

$du=2t\,dt$

$a=\tan u$

$da=\sec^2 u\,du$

55. $f_{ave} = \frac{1}{\pi-(-\pi)} \int_{-\pi}^{\pi} \sin^2 x \cos^3 x \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x (1-\sin^2 x) \cos x \, dx = \frac{1}{2\pi} \int_0^0 [u^2 - u^4] \, du = 0.$

$u = \sin x, \ du = \cos x \, dx$

59. $\int_0^{2\pi} \cos^3 x \, dx = \int_0^{2\pi} (1-\sin^2 x) \cos x \, dx = \int_0^0 [1-u^2] \, du = 0.$

$u = \sin x, \ du = \cos x \, dx$

61. $y = \sin x, \ y = 0, \ \frac{\pi}{2} \le x \le \pi$; about the x axis $V = \int_{x_1}^{x_2} A(x) \, dx = \int_{\frac{\pi}{2}}^{\pi} \pi \sin^2 x \, dx = \pi \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}[1-\cos 2x] \, dx$

$A(x) = \pi r^2 = \pi \sin^2 x$

$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{2}}^{\pi} = \frac{\pi}{2}\left[(\pi - 0) - \left(\frac{\pi}{2} - 0\right)\right] = \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \boxed{\frac{\pi^2}{4}}$