7.1 homework Integration by parts

1. $\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\frac{x^3}{3} + C = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C.$

$= \frac{1}{9}x^3[3\ln x - 1] + C.$

$u = \ln x \quad v = \frac{x^3}{3}$

$du = \frac{1}{x}dx \quad dv = x^2 dx$

3. $\int x\cos 5x\, dx = \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\, dx = \frac{1}{5}x\sin 5x - \frac{1}{5}\left[\frac{-\cos 5x}{5}\right] = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C.$

$u = x \quad v = \frac{\sin 5x}{5}$

$du = dx \quad dv = \cos 5x\, dx$

5. $\int r e^{\frac{r}{2}} dr = 2re^{\frac{1}{2}r} - \int 2e^{\frac{1}{2}r}dr = 2re^{\frac{1}{2}r} - 2\cdot 2e^{\frac{1}{2}r} + C = 2re^{\frac{1}{2}r} - 4e^{\frac{1}{2}r} + C.$

$= 2e^{\frac{r}{2}}(r-2) + C.$

$u = r \quad v = 2e^{\frac{1}{2}r}$

$du = dr \quad dv = e^{\frac{1}{2}r}dr$

7. $\int x^2 \sin \pi x\, dx = -\frac{1}{\pi}x^2\cos\pi x + \int \frac{2}{\pi}x\cos\pi x\, dx = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\left[\frac{1}{\pi}x\sin\pi x - \int\frac{1}{\pi}\sin\pi x\, dx\right]$

$\boxed{u = x^2 \quad v = \frac{-\cos\pi x}{\pi} \quad du = 2x\,dx \quad dv = \sin\pi x\, dx}$

$\boxed{U = x \quad V = \frac{\sin\pi x}{\pi} \quad dU = dx \quad dV = \cos\pi x}$

$= -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x - \frac{2}{\pi^2}\left[\frac{-\cos\pi x}{\pi}\right] + C = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x + \frac{2}{\pi^3}\cos\pi x + C.$

9. $\int \ln(2x+1)\, dx = x\ln(2x+1) - \frac{1}{2}\int\frac{2x}{2x+1}dx \overset{(2)}{=} x\ln(2x+1) - \frac{1}{2}\int\frac{u-1}{u}du = x\ln(2x+1) - \frac{1}{2}\int\left[1-\frac{1}{u}\right]du$

$= x\ln(2x+1) - \frac{1}{2}[u - \ln|u|] + C$

$= x\ln(2x+1) - \frac{1}{2}(2x+1) + \frac{1}{2}\ln|2x+1| + C$

$= x\ln(2x+1) - x - \frac{1}{2} + \frac{1}{2}\ln|2x+1| + C$

$= x\ln(2x+1) - x + \frac{1}{2}\ln|2x+1| + C.$

$\boxed{u = \ln(2x+1) \quad v = x \quad du = \frac{1}{2x+1}\cdot 2dx \quad dv = dx}$

$\boxed{u = 2x+1 \quad u-1 = 2x \quad du = 2dx}$

11. $\int \arctan 4t\, dt = t\arctan 4t - \frac{1}{8}\int\frac{4t\cdot 8}{1+16t^2}dt = t\arctan 4t - \frac{1}{8}\int\frac{du}{u} = t\arctan 4t - \frac{1}{8}\ln|u| + C$

$= t\arctan 4t - \frac{1}{8}\ln(1+16t^2) + C$

$\boxed{u = \arctan 4t \quad v = t \quad du = \frac{1}{1+(4t)^2}\cdot 4dt \quad dv = dt}$

$\boxed{U = 1+16t^2 \quad dU = 32t\, dt}$

13. $\int t\sec^2 2t\,dt = \frac{1}{2}t\cdot\tan 2t - \frac{1}{2}\int\frac{1}{2}\tan 2t\,dt^{\cdot 2} = \frac{1}{2}t\cdot\tan 2t - \frac{1}{4}\int\tan U\,dU = \frac{1}{2}t\cdot\tan 2t - \frac{1}{4}\left[-\ln|\cos U|\right]+C$

$\boxed{\begin{array}{l} U=2t \\ dU=2dt \end{array}}$

$u=t \quad v=\frac{1}{2}\tan 2t$

$du=dt \quad dv=\sec^2 2t\,dt$

$= \frac{1}{2}t\cdot\tan 2t + \frac{1}{4}\ln|\cos 2t| + C.$

$\left(\text{OR } \frac{1}{2}t\cdot\tan 2t - \frac{1}{4}\ln|\sec 2t| + C\right)$

$\frac{1}{2}\int\sec^2 2t\,dt^{(2)} = \frac{1}{2}\int\sec^2 x\,dx = \frac{1}{2}\tan x = \frac{1}{2}\tan 2t$

$x=2t$

$dx=2dt$

15. $\int(\ln x)^2dx = x(\ln x)^2 - \int 2\ln x\,dx = x(\ln x)^2 - 2\left[x\ln x - \int dx\right] = x(\ln x)^2 - 2x\ln x + 2x + C.$

$\boxed{\begin{array}{ll} u=(\ln x)^2 & v=x \\ du=\frac{2\ln x}{x}dx & dv=dx \end{array}}$ $\boxed{\begin{array}{ll} U=\ln x & V=x \\ dU=\frac{1}{x}dx & dv=dx \end{array}}$

17. $\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \int\frac{3}{2}e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\left[\frac{1}{2}e^{2\theta}\cos 3\theta + \int\frac{3}{2}e^{2\theta}\sin 3\theta\,d\theta\right]$

$\boxed{\begin{array}{ll} u=\sin 3\theta & v=\frac{e^{2\theta}}{2} \\ du=3\cos 3\theta\,d\theta & dv=e^{2\theta}d\theta \end{array}}$ $\boxed{\begin{array}{ll} U=\cos 3\theta & V=\frac{1}{2}e^{2\theta} \\ dU=-3\sin 3\theta\,d\theta & dv=e^{2\theta}d\theta \end{array}}$

So: $\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta$ $\qquad 1+\frac{9}{4}=\frac{13}{4}$

$\Rightarrow \frac{13}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta \Rightarrow \int e^{2\theta}\sin 3\theta\,d\theta = \frac{\cancel{4}^{2}}{13}\cdot\frac{1}{\cancel{2}}e^{2\theta}\sin 3\theta - \frac{\cancel{4}}{13}\cdot\frac{3}{\cancel{4}}e^{2\theta}\cos 3\theta + C$

$= \frac{2}{13}e^{2\theta}\sin 3\theta - \frac{3}{13}e^{2\theta}\cos 3\theta + C$

$= \frac{1}{13}e^{2\theta}\left[2\sin 3\theta - 3\cos 3\theta\right] + C.$

19. $\int_0^\pi t\sin 3t\,dt = -\frac{1}{3}t\cos 3t + \int\frac{1}{3}\cos 3t\,dt = \left[-\frac{1}{3}t\cos 3t + \frac{1}{3}\cdot\frac{\sin 3t}{3}\right]_0^\pi$

$= \left[-\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t\right]_0^\pi$

$u=t \quad V=\frac{-\cos 3t}{3}$

$du=dt \quad dv=\sin 3t\,dt$

$= \left[-\frac{\pi}{3}\cos 3\pi + \frac{1}{9}\sin 3\pi\right] - \left[0+0\right] = -\frac{\pi}{3}(-1) + \frac{1}{9}(0) = \frac{\pi}{3}.$

21. $\int_0^1 t\cosh t\,dt = t\sinh t - \int\sinh t\,dt = \left[t\sinh t - \cosh t\right]_0^1$ $\qquad \sinh x = \frac{e^x-e^{-x}}{2}$

$= (\sinh 1 - \cosh 1) - (0 - \cosh 0)$ $\qquad \cosh x = \frac{e^x+e^{-x}}{2}$

$u=t \quad v=\sinh t$

$du=dt \quad dv=\cosh t\,dt$

$= \frac{e^1-e^{-1}}{2} - \frac{e^1+e^{-1}}{2} + \frac{e^0+e^0}{2}$

$= \frac{\cancel{e}-e^{-1}-\cancel{e}-e^{-1}+1+1}{2} = \frac{-2e^{-1}+2}{2} = 1-e^{-1} = 1-\frac{1}{e}.$

23. $\int_1^2 \frac{\ln x}{x^2}dx = -\frac{1}{x}\ln x + \int x^{-2}dx = -\frac{1}{x}\ln x + \frac{x^{-1}}{-1} = \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_1^2 = \left(-\frac{1}{2}\ln 2 - \frac{1}{2}\right) - (0-1)$

$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$

$= \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}[1-\ln 2].$

$u = \ln x \quad v = \frac{x^{-1}}{-1} = \frac{-1}{x}$

$du = \frac{1}{x}dx \quad dv = x^{-2}dx$

25. $\int_0^1 \frac{y}{e^{2y}}dy = \int_0^1 ye^{-2y}dy = -\frac{1}{2}ye^{-2y} + \int \frac{1}{2}e^{-2y}dy = -\frac{1}{2}ye^{-2y} + \frac{1}{2}\frac{e^{-2y}}{-2} = \left[-\frac{1}{2}ye^{-2y} - \frac{1}{4}e^{-2y}\right]_0^1$

$u = y \quad v = \frac{e^{-2y}}{-2}$

$du = dy \quad dv = e^{-2y}dy$

$= \left(-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right) - \left(0 - \frac{1}{4}e^0\right) = -\frac{3}{4}e^{-2} + \frac{1}{4} = \frac{1}{4} - \frac{3}{4e^2} = \frac{1}{4}\left[1 - \frac{3}{e^2}\right].$

$= \frac{1}{4e^2}\left[e^2 - 3\right].$

27. $\int_0^{\frac{1}{2}} \cos^{-1}x\,dx = x\cos^{-1}x + \left(\frac{-1}{2}\right)\int \frac{x(-2)}{\sqrt{1-x^2}}dx = x\cos^{-1}x - \frac{1}{2}\int u^{-\frac{1}{2}}du = x\cos^{-1}x - \frac{1}{2}\left[2u^{\frac{1}{2}}\right]$

$\boxed{u = \cos^{-1}x \quad v = x \quad du = \frac{-1}{\sqrt{1-x^2}}dx \quad dv = dx}$

$\boxed{U = 1-x^2 \quad dU = -2x\,dx}$

$= \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{2}\cdot\frac{\pi}{3} - \sqrt{\frac{3}{4}}\right) - (0-1) = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1. = \frac{1}{6}\left[\pi - 3\sqrt{3} + 6\right]$

29. $\int \cos x \ln(\sin x)\,dx = \sin x\cdot\ln(\sin x) - \int \cos x\,dx = \sin x\cdot\ln(\sin x) - \sin x + C = \sin x\left[\ln(\sin x) - 1\right] + C$

$u = \ln(\sin x) \quad v = \sin x$

$du = \frac{1}{\sin x}\cdot\cos x\,dx \quad dv = \cos x\,dx$

31. $\int_1^2 x^4(\ln x)^2dx = \frac{1}{5}x^5(\ln x)^2 - \int \frac{2}{5}x^4\ln x\,dx = \frac{1}{5}x^5(\ln x)^2 - \frac{2}{5}\left[\frac{1}{5}x^5\ln x - \int \frac{1}{5}x^4dx\right]$

$\boxed{u = (\ln x)^2 \quad v = \frac{x^5}{5} \quad du = \frac{2\ln x}{x}dx \quad dv = x^4dx}$

$\boxed{U = \ln x \quad V = \frac{x^5}{5} \quad dU = \frac{1}{x}dx \quad dV = x^4dx}$

$= \left[\frac{1}{5}x^5(\ln x)^2 - \frac{2}{25}x^5\ln x + \frac{2}{25}\frac{x^5}{5}\right]_1^2$

$= \left[\frac{x^5}{125}\left(25(\ln x)^2 - 10\ln x + 2\right)\right]_1^2$

$= \frac{32}{125}\left(25(\ln 2)^2 - 10\ln 2 + 2\right) - \frac{1}{125}(0-0+2) = \frac{32}{125}\left(25(\ln 2)^2 - 10\ln 2\right) + \frac{62}{125}$ OR $\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}.$

33. $\int \cos\sqrt{x}\,dx = \int \cos t\cdot 2t\,dt = 2\int t\cos t\,dt = 2\left[t\sin t - \int \sin t\,dt\right] = 2t\sin t + 2\cos t + C$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$

$u = t \quad v = \sin t$

$du = dt \quad dv = \cos t\,dt$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}dx$

$2\sqrt{x}\,dt = dx$

$2t\,dt = dx$

35. $\frac{1}{2}\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta \cdot (2) = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} t\cos t\, dt = \frac{1}{2}\left[t\sin t - \int \sin t\, dt\right] = \left[\frac{1}{2}t\sin t + \frac{1}{2}\cos t\right]_{\frac{\pi}{2}}^{\pi}$

$t = \theta^2$

$dt = 2\theta\, d\theta$

$u = t \quad v = \sin t$

$du = dt \quad dv = \cos t\, dt$

$= \left[\left(0 + \frac{1}{2}(-1)\right) - \left(\frac{\pi}{4}(1) + 0\right)\right] = -\frac{1}{2} - \frac{\pi}{4}.$

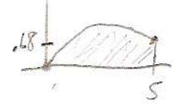
47. $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$ Proof: $u = (\ln x)^n \quad v = x$

$du = \frac{n(\ln x)^{n-1}}{x}\, dx \quad dv = dx$

So: $\int (\ln x)^n dx = x(\ln x)^n - \int x \cdot \frac{n(\ln x)^{n-1}}{x}\, dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx.$

53. Area bounded by $y = xe^{-0.4x}$, $y = 0$, $x = 5$.

x	y
0	0
5	$\frac{5}{e^2} = .68$



$\int_0^5 xe^{-0.4x} dx$

$u = x \quad v = \frac{e^{-0.4x}}{-0.4}$

$du = dx \quad dv = e^{-0.4x} dx$

$= -\frac{1}{0.4}xe^{-0.4x} + \frac{1}{0.4}\int e^{-0.4x} dx = -\frac{5}{2}xe^{-0.4x} + \frac{5}{2}\frac{e^{-0.4x}}{-0.4} = \left[-\frac{5}{2}xe^{-0.4x} - \frac{25}{4}e^{-0.4x}\right]_0^5$

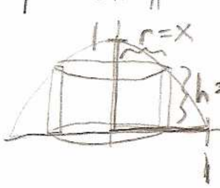
$= \left(-\frac{25}{2}e^{-2} - \frac{25}{4}e^{-2}\right) - \left(0 - \frac{25}{4}\right) = -\frac{25}{2}e^{-2} - \frac{25}{4}e^{-2} + \frac{25}{4} = -\frac{75}{4}e^{-2} + \frac{25}{4}.$

57. $y = \cos\left(\frac{\pi x}{2}\right)$, $y = 0$, $0 \le x \le 1$; about the y axis $V = \int_{x_1}^{x_2} 2\pi r h(x)\, dx = \int_0^1 2\pi x \cos\left(\frac{\pi x}{2}\right) dx$

per $= 2\pi \cdot \frac{2}{\pi} = 4$

$u = x \quad v = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)$

$du = dx \quad dv = \cos\left(\frac{\pi x}{2}\right) dx$



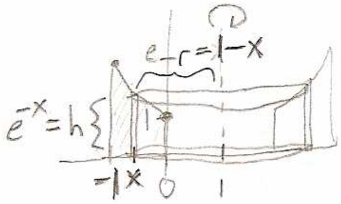
$2\pi \cdot \left(\left[\frac{2}{\pi}x\sin\frac{\pi x}{2}\right]_0^1 - \int_0^1 \frac{2}{\pi}\sin\frac{\pi x}{2}\, dx\right)$

$2\pi \cdot \left(\left(\frac{2}{\pi} - 0\right) + \left[\frac{2}{\pi}\cos\frac{\pi x}{2} \cdot \frac{2}{\pi}\right]_0^1\right)$

$= 2\pi \cdot \left(\frac{2}{\pi} + \frac{4}{\pi^2}(0-1)\right) = 4 - \frac{8}{\pi}.$

59. $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$; about $x = 1$ $V = \int_{x_1}^{x_2} 2\pi r h(x)\, dx = \int_{-1}^{0} 2\pi(1-x)e^{-x} dx = 2\pi\int_{-1}^{0} e^{-x} dx - 2\pi\int_{-1}^{0} xe^{-x} dx$

$\boxed{u = x \quad v = -e^{-x} \\ du = dx \quad dv = e^{-x} dx}$



$= 2\pi\left[-e^{-x}\right]_{-1}^{0} - 2\pi\left[-xe^{-x} + \int e^{-x} dx\right]$

$= -2\pi(1-e) + 2\pi\left[xe^{-x}\right]_{-1}^{0} + 2\pi\left[e^{-x}\right]_{-1}^{0}$

$= \cancel{-2\pi} + \cancel{2\pi e} + 2\pi[0+e] + 2\pi[\cancel{1-e}] = 2\pi e.$

61. $f(x) = x^2\ln x$ on $[1,3]$. $f_{ave} = \frac{1}{3-1}\int_1^3 x^2\ln x\, dx = \frac{1}{2}\left[\frac{1}{3}x^3\ln x - \int \frac{1}{3}x^2 dx\right] = \left[\frac{1}{6}x^3\ln x - \frac{1}{18}x^3\right]_1^3$

$u = \ln x \quad v = \frac{x^3}{3}$

$du = \frac{1}{x}dx \quad dv = x^2 dx$

$= \left(\frac{27}{6}\ln 3 - \frac{27}{18}\right) - \left(0 - \frac{1}{18}\right) = \frac{9}{2}\ln 3 - \frac{26}{18} = \frac{9}{2}\ln 3 - \frac{13}{9}.$