1. Find the work done in lifting a 40kg sandbag to a height of 1.5m.

$W = \text{Force} \cdot \text{distance} = \text{mass} \cdot \text{acc} \cdot \text{dist.} = mgd = 40\text{kg} \cdot 9.8\frac{m}{s^2} \cdot 1.5m = \boxed{588\,J}$ ← joules

3. A particle is moved along the x-axis by a force that measures $\frac{10}{(1+x)^2}$ lbs at a point x feet from the origin. Find the work done in moving the particle from the origin to a distance of 9 ft.

For a variable force, $W = \int_a^b f(x)\,dx$, so $W = \int_0^9 \frac{10}{(1+x)^2}dx = 10\int_1^{10} u^{-2}\,du = 10\left[\frac{-1}{u}\right]_1^{10}$

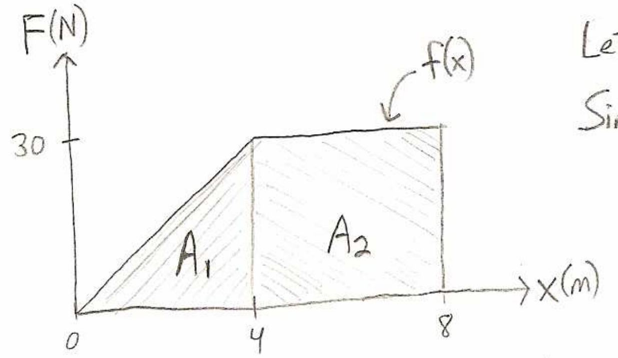
↑ f(x)

$u = 1+x$
$du = dx$

x	u
0	1
9	10

$= -10\left[\frac{1}{10} - 1\right] = -10\left[-\frac{9}{10}\right] = \boxed{9\text{ ft-lb.}}$

5. Shown is the graph of a force function (in newtons) that increases to its maximum value and then remains constant. How much work is done in moving an object a distance of 8m?



Let f(x) represent the force function.

Since the area under the curve $= \int_0^8 f(x)\,dx$, and because this is a variable force $W = \int_0^8 f(x)\,dx$,

the work done = the area under the curve $= A_1 + A_2$

$A_1 = \frac{1}{2} \cdot 4 \cdot 30 = 60$. $A_2 = 4 \cdot 30 = 120$. So, $W = 60 + 120 = \boxed{180\,J}$

7. A force of 10 lb is required to hold a spring stretched $\underbrace{4\text{-in.}}_{\frac{1}{3}ft}$ beyond its natural length. How much work is done in stretching it from its natural length to 6in ($\frac{1}{2}$ ft) beyond its natural length?

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is $f(x) = kx$.

Using $f(x) = kx$, we have $10 = k \cdot \frac{1}{3} \Rightarrow 30 = k$, so $f(x) = 30x$.

$W = \int f(x)\,dx = \int_0^{\frac{1}{2}} 30x\,dx = 30\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} = 15\left[\left(\frac{1}{2}\right)^2\right] = \boxed{\frac{15}{4}\text{ ft-lb}}$

9. Suppose that 2J of work is needed to stretch a spring from its natural length of 30 cm (.30m) to a length of 42cm (.42m).

a) How much work is needed to stretch the spring from 35cm (.35m) to 40cm (.40m)?

2J is needed to stretch the spring from its natural length to .12m beyond its natural length.

So: $W = \int_a^b f(x)\,dx \Rightarrow 2 = \int_0^{.12} kx\,dx \Rightarrow 2 = k\left[\frac{x^2}{2}\right]_0^{.12} \Rightarrow 2 = \frac{k}{2}\cdot(.12)^2 \Rightarrow k = \frac{4}{(.12)^2} \Rightarrow$

$k \approx 277.\bar{7}$

So $f(x) = 277.\bar{7}x$.

In part a, the spring is being stretched from .35m, which is .05m beyond its natural length, to .40m, which is .10m beyond its natural length. So, $W = \int_{.05}^{.10} f(x)\,dx$ = (next line)

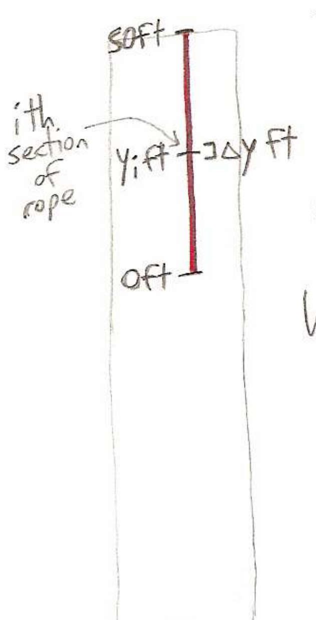
$$= \int_{.05}^{.10} 277.\bar{7}x\,dx = 277.\bar{7}\left[\frac{x^2}{2}\right]_{.05}^{.10} = \frac{277.\bar{7}}{2}\left[.10^2 - .05^2\right] = \frac{277.\bar{7}}{2}\left[.0075\right] \approx \boxed{1.04\text{ J}}$$

b) How far beyond its natural length will a force of 30N keep the spring stretched?

$f(x) = 277.\bar{7}x$, so $30 = 277.\bar{7}x \Rightarrow x = \frac{30}{277.\bar{7}} \Rightarrow x = \boxed{.108\text{m OR } 10.8\text{cm}}$

13. A heavy rope, 50ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120ft high.

a) How much work is done in pulling the rope to the top of the building?



The ith section of rope weighs $0.5\frac{\text{lb}}{\cancel{\text{ft}}}\cdot\Delta y\,\cancel{\text{ft}} = 0.5\Delta y\text{ lb} = F_i$ (Force required to raise the ith section).

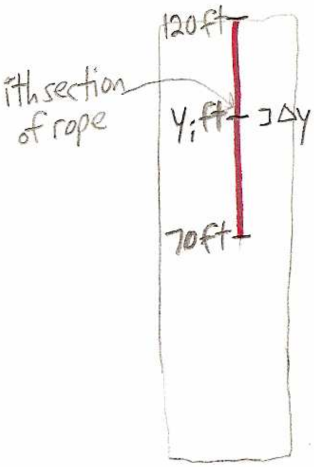
d_i = the distance the ith section must be raised = $(50 - y_i)$ ft.

W_i = the work required to raise the ith section = $F_i d_i = 0.5(50 - y_i)\Delta y$ ft-lb.

W = the total work done in pulling the entire rope up = $\lim_{n\to\infty}\sum_{i=1}^{n} 0.5(50 - y_i)\Delta y$ = (next line)

$$= \int_0^{50} 0.5(50 - y)\,dy = 0.5\left[50y - \frac{y^2}{2}\right]_0^{50} = 0.5\left[2500 - 1250\right] = \boxed{625\text{ ft-lb}}$$

13a. Here is another way to do 13a: (same procedure, but with a different way of labeling the vertical axis)



The ith section of rope weighs $0.5 \frac{lb}{\cancel{ft}} \cdot \Delta y \cancel{ft} = 0.5 \Delta y \text{ lb} = F_i$

$d_i = (120 - y_i)$ ft.

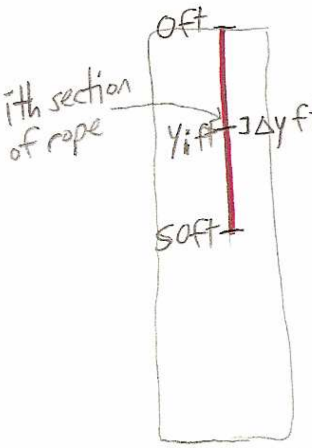
$W_i = F_i d_i = 0.5(120 - y_i)\Delta y$ ft-lb.

$W = \text{total work} = \lim_{n\to\infty} \sum_{i=1}^{n} 0.5(120 - y_i)\Delta y = \int_{70}^{120} 0.5(120 - y)\,dy$

$= 0.5\left[120y - \frac{y^2}{2}\right]_{70}^{120} = 0.5\left[(14{,}400 - 7200) - (8400 - 2450)\right]$

$= 0.5\left[7200 - 5950\right] = \boxed{625 \text{ ft-lb}}$

13a. Yet another way to do 13a: (same procedure again, but with a different way of labeling the vertical axis)



the ith section of rope weighs $0.5 \frac{lb}{\cancel{ft}} \cdot \Delta y \cancel{ft} = 0.5 \Delta y \text{ lb} = F_i$

$d_i = y_i$ ft.

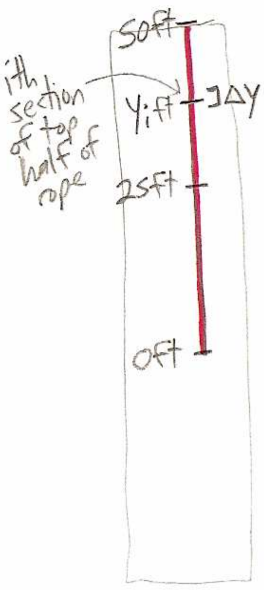
$W_i = F_i d_i = 0.5 y_i \Delta y$ ft-lb.

$W = \text{total work} = \lim_{n\to\infty} \sum_{i=1}^{n} 0.5 y_i \Delta y = \int_0^{50} 0.5 y\,dy = 0.5\left[\frac{y^2}{2}\right]_0^{50}$

$= \frac{0.5}{2}\left[50^2\right] = \boxed{625 \text{ ft-lb.}}$

13b. How much work is done in pulling half of the rope to the top of the building?

The total work $W = W_1 + W_2$, where W_1 = work done raising the top half all the way up
W_2 = work done raising the bottom half 25 ft.



- Let's focus first on the top half: The ith section of the top half weighs $0.5 \frac{lb}{\cancel{ft}} \cdot \Delta y \cancel{ft} = \underbrace{0.5 \Delta y \text{ lb}}_{F_i}$

  $d_i = (50 - y_i)$ ft. $W_i = F_i d_i = 0.5(50 - y_i)\Delta y$ ft-lb.

  $W_1 = \lim_{n\to\infty} \sum_{i=1}^{n} 0.5(50 - y_i)\Delta y = \int_{25}^{50} 0.5(50 - y)\,dy = 0.5\left[50y - \frac{y^2}{2}\right]_{25}^{50}$ = (next line)

  $= 0.5\left[(2500 - 1250) - (1250 - 312.5)\right] = 0.5\left[1250 - 937.5\right] = 156.25$ ft-lb. $= W_1$.

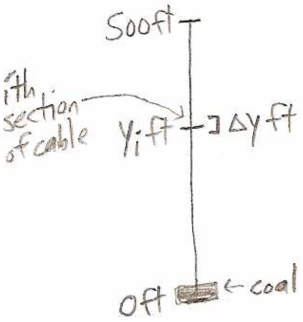
- Now the bottom half: The ith section weighs $0.5\Delta y \text{ lb} = F_i$. $d_i = 25$ ft.

  $W_i = F_i d_i = 12.5 \Delta y$ ft-lb. $W = \lim_{n\to\infty} \sum_{i=1}^{n} 12.5 \Delta y = \int_0^{25} 12.5\,dy = 12.5y\Big]_0^{25} = \boxed{\underbrace{312.5 \text{ ft-lb}}_{W_2}}$

- So: $W = W_1 + W_2 = 156.25 + 312.5 = \boxed{468.75 \text{ ft-lb} = W}$

13b. Here's another way to get W_2: The entire bottom half of the rope weighs $0.5\frac{lb}{ft}\cdot 25ft = 12.5\,lb\,(F)$ and must be raised $25ft\,(d)$. So, $W_2 = Fd = (12.5)(25) = 312.5$ ft-lb.

15. A cable that weighs 2 lb/ft is used to lift 800 lb. of coal up a mine shaft 500 ft deep. Find the work done.



Total work done $= W_1 + W_2$, where W_1 = work done lifting the cable
W_2 = work done lifting the coal 500 ft.

- To get W_1: The ith section of cable weighs $2\frac{lb}{ft}\cdot \Delta y\, \cancel{ft} = 2\Delta y\, lb = F_i$ (force required to raise the ith section)

  d_i = distance the ith section must be raised $= (500-y_i)$ ft.

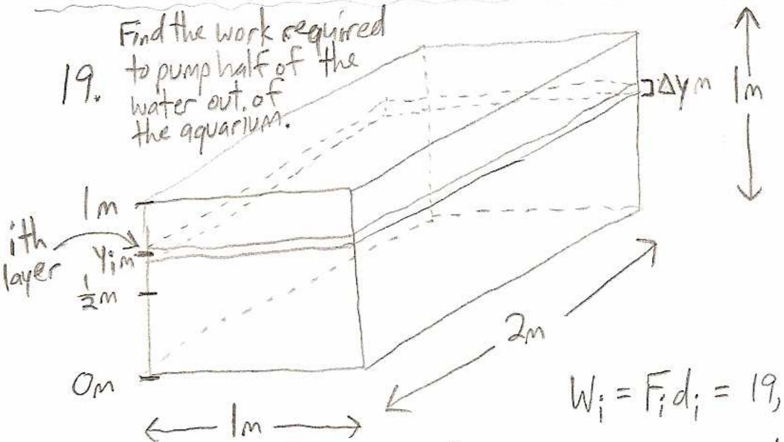
  W_i = work required to raise the ith section $= F_i d_i = 2(500-y_i)\Delta y$ ft-lb.

  $$W_1 = \lim_{n\to\infty}\sum_{i=1}^{n} 2(500-y_i)\Delta y = \int_0^{500} 2(500-y)\,dy = 2\left[500y - \frac{y^2}{2}\right]_0^{500} = 2[250{,}000 - 125{,}000]$$
  $$= \boxed{250{,}000 \text{ ft-lb.} = W_1}$$

- To get W_2: The coal weighs 800 lb = the force required to lift it. It must be raised 500 ft = d, so $W_2 = Fd = 800\cdot 500 = \boxed{400{,}000 \text{ ft-lb} = W_2}$

- So: $W = W_1 + W_2 = 250{,}000 + 400{,}000 = \boxed{650{,}000 \text{ ft-lb} = W}$

19. Find the work required to pump half of the water out of the aquarium.



V_i = volume of ith layer $= LWH = 2\cdot 1\cdot \Delta y = 2\Delta y\, m^3$.

mass of ith layer $= 2\Delta y\, m^3 \cdot 1000\frac{kg}{m^3} = 2000\Delta y$ kg.

F_i = force required to raise ith layer $= (2000\Delta y\, kg)(9.8\frac{m}{s^2}) = 19{,}600\Delta y$ N.

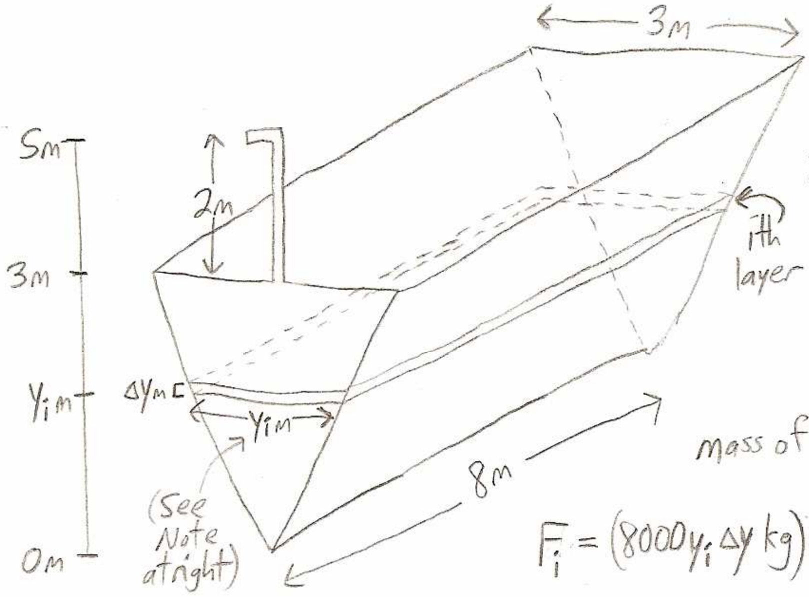
d_i = distance the ith layer must be raised $= (1-y_i)$ m.

$W_i = F_i d_i = 19{,}600(1-y_i)\Delta y$ J = work done raising the ith layer.

$$\text{Total work} = W = \lim_{n\to\infty}\sum_{i=1}^{n} 19{,}600(1-y_i)\Delta y = \int_{\frac{1}{2}}^{1} 19{,}600(1-y)\,dy = 19{,}600\left[y - \frac{y^2}{2}\right]_{\frac{1}{2}}^{1}$$

$$= 19{,}600\left[\left(1-\tfrac{1}{2}\right) - \left(\tfrac{1}{2}-\tfrac{1}{8}\right)\right] = 19{,}600\left[\tfrac{1}{2}-\tfrac{3}{8}\right] = 19{,}600\left[\tfrac{1}{8}\right] = \boxed{2450\,J}$$

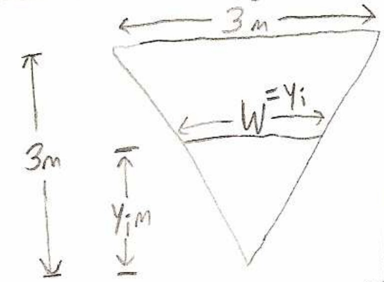
21. Find the work required to pump the water out of the spout. (The tank is full of water.)



$V_i = LWH = 8 y_i \Delta y \ m^3$ = volume of ith layer.

Note: $W = y_i$ because of similar trianges:

$\frac{3}{3} = \frac{W}{y_i}$ thus, $W = y_i$.



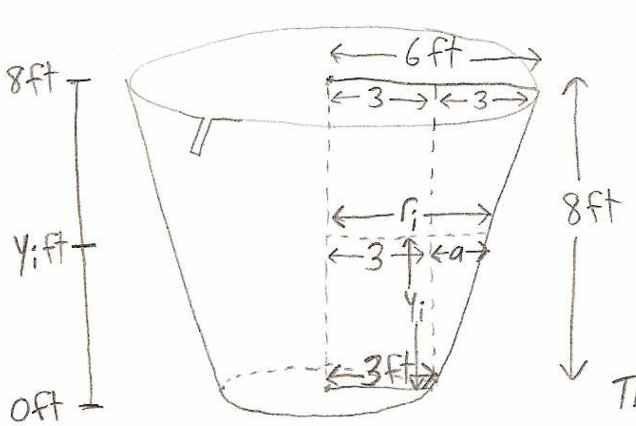
mass of ith layer $= 8 y_i \Delta y \ m^3 \cdot 1000 \frac{kg}{m^3} = 8000 y_i \Delta y$ kg.

$F_i = (8000 y_i \Delta y \text{ kg})(9.8 \frac{m}{s^2}) = 78{,}400 y_i \Delta y$ N. $d_i = (5 - y_i)$ m.

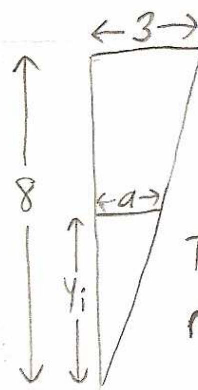
$W_i = F_i d_i = 78{,}400(5y_i - y_i^2)\Delta y$ J. Total work $= W = \lim_{n\to\infty} \sum_{i=1}^{n} 78{,}400(5y_i - y_i^2)\Delta y$ = (next line)

$W = \int_0^3 78{,}400(5y - y^2)\,dy = 78{,}400\left[\frac{5}{2}y^2 - \frac{1}{3}y^3\right]_0^3 = 78{,}400\left[\frac{45}{2} - 9\right] = 78{,}400\left[\frac{27}{2}\right] = \boxed{1{,}058{,}400 \text{ J}}$

23. The tank is full of water. Find the work required to pump the water out of the spout.



By similar triangles:



$\frac{3}{8} = \frac{a}{y_i}$

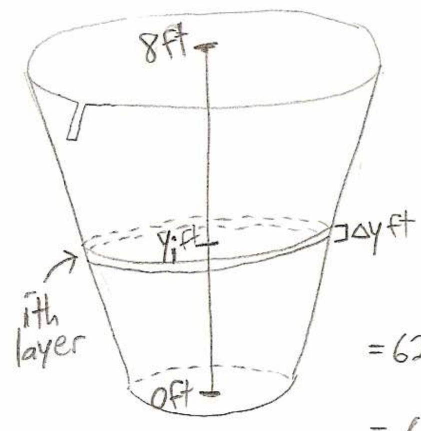
so, $a = \frac{3}{8} y_i$ ft.

Thus, since $r_i = 3 + a$,

$r_i = (3 + \frac{3}{8} y_i)$ ft.

The ith layer is a disk with volume $V_i = \pi r_i^2 h = \pi (3 + \frac{3}{8} y_i)^2 \Delta y \ ft^3$.

The ith layer weighs $\left[\pi (3 + \frac{3}{8} y_i)^2 \Delta y \ \cancel{ft^3}\right]\left[62.5 \frac{lb}{\cancel{ft^3}}\right] = 62.5\pi (3 + \frac{3}{8} y_i)^2 \Delta y$ lb $= F_i$.



$d_i = (8 - y_i)$ ft. $W_i = F_i d_i = 62.5\pi (3 + \frac{3}{8} y_i)^2 (8 - y_i)\Delta y$ ft-lb.

$W = \int_0^8 62.5\pi (3 + \frac{3}{8} y)^2 (8 - y)\,dy = 62.5\pi \int_0^8 \left(9 + \frac{9}{4} y + \frac{9}{64} y^2\right)(8 - y)\,dy$

$= 62.5\pi \int_0^8 \left[72 - 9y + 18y - \frac{9}{4} y^2 + \frac{9}{8} y^2 - \frac{9}{64} y^3\right] dy$

$= 62.5\pi \int_0^8 \left[72 + 9y - \frac{9}{8} y^2 - \frac{9}{64} y^3\right] dy = 62.5\pi \left[72y + \frac{9}{2} y^2 - \frac{3}{8} y^3 - \frac{9}{256} y^4\right]_0^8$

$= 62.5\pi\left[576 + 288 - 192 - 144\right] = 62.5\pi\left[528\right] \approx \boxed{103{,}673 \text{ ft-lb}}$