6.1 Areas Between Curves

1. $A = \int_0^4 (5x - x^2 - x)\,dx = \int_0^4 (4x - x^2)\,dx = \left[\frac{\cancel{4}^{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{96-64}{3} = \boxed{\frac{32}{3}}$

3. $A = \int_{-1}^{1} [e^y - (y^2 - 2)]\,dy = \left[e^y - \frac{y^3}{3} + 2y\right]_{-1}^{1} = (e - \frac{1}{3} + 2) - (\frac{1}{e} + \frac{1}{3} - 2)$

$= e - \frac{1}{3} + 2 - \frac{1}{e} - \frac{1}{3} + 2 = e - \frac{1}{e} + 4 - \frac{2}{3} = \boxed{e - \frac{1}{e} + \frac{10}{3}}$

5. $y = x+1,\ y = 9 - x^2,\ x = -1,\ x = 2.$ Curves intersect when $x + 1 = 9 - x^2 \Rightarrow x^2 + x - 8 = 0$

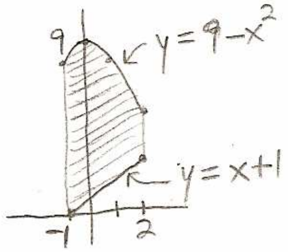
$\Rightarrow x = \frac{-1 \pm \sqrt{1 - 4\cdot 1\cdot(-8)}}{2} = \frac{-1 \pm \sqrt{33}}{2} \approx 2.37$ and $-3.37.$

$y = x+1$

x	y
-1	0
2	3

$y = 9 - x^2$

x	y
-1	8
2	5



$A = \int_{-1}^{2} [(9 - x^2) - (x+1)]\,dx$

$= \int_{-1}^{2} [8 - x^2 - x]\,dx$

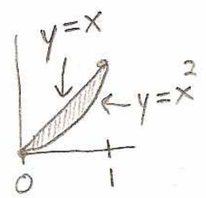
$= \left[8x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{2} = (16 - \frac{8}{3} - 2) - (-8 + \frac{1}{3} - \frac{1}{2})$

$= 14 - \frac{8}{3} + 8 - \frac{1}{3} + \frac{1}{2} = 22 - 3 + \frac{1}{2} = 19\frac{1}{2} = \boxed{\frac{39}{2}}$

7. $y = x,\ y = x^2$ Curves intersect when $x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1$

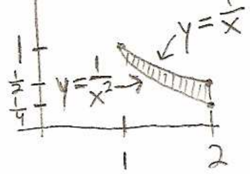
x	y
0	0
1	1

x	y
0	0
1	1



$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$

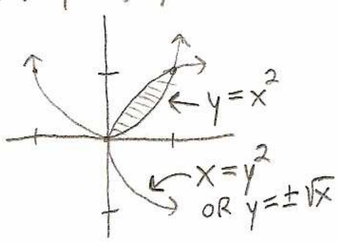
9. $y = \frac{1}{x},\ y = \frac{1}{x^2},\ x = 2$ Curves intersect when $\frac{1}{x} = \frac{1}{x^2} \Rightarrow x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$
$\Rightarrow \cancel{x=0}, x = 1.$

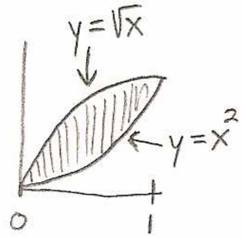


$A = \int_1^2 \left[\frac{1}{x} - \frac{1}{x^2}\right] dx = \int_1^2 \left[\frac{1}{x} - x^{-2}\right] dx = \left[\ln|x| - \frac{x^{-1}}{-1}\right]_1^2$

$= \left[\ln|x| + \frac{1}{x}\right]_1^2 = (\ln 2 + \frac{1}{2}) - (0 + 1) = \boxed{\ln 2 - \frac{1}{2}}$

11. $y = x^2,\ y^2 = x$





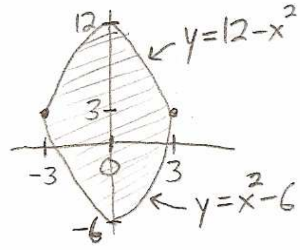
$A = \int_0^1 [\sqrt{x} - x^2]\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$

13. $y = 12 - x^2$, $y = x^2 - 6$ Curves intersect when $12 - x^2 = x^2 - 6 \Rightarrow 18 = 2x^2 \Rightarrow 9 = x^2 \Rightarrow x = \pm 3$.

x	y
-3	3
3	3

x	y
-3	3
3	3



By symmetry, $A = 2 \cdot \int_0^3 \left[(12 - x^2) - (x^2 - 6)\right] dx$

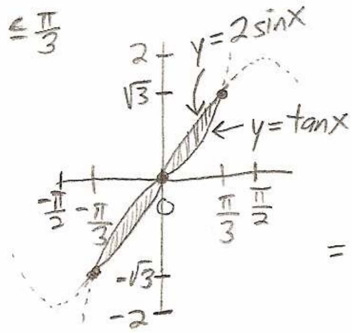
$= 2 \cdot \int_0^3 \left[18 - 2x^2\right] dx = 2\left[18x - \frac{2}{3}x^3\right]_0^3$

$= 2\left[54 - 18\right] = 2[36] = \boxed{72}$

15. $y = \tan x$, $y = 2\sin x$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

x	y
$-\frac{\pi}{3}$	$-\sqrt{3}$
0	0
$\frac{\pi}{3}$	$\sqrt{3}$

x	y
$-\frac{\pi}{3}$	$2 \cdot \left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$
0	0
$\frac{\pi}{3}$	$2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$



By symmetry, $A = 2 \cdot \int_0^{\frac{\pi}{3}} \left[2\sin x - \tan x\right] dx$

$= 2 \cdot \left[-2\cos x - (-\ln|\cos x|)\right]_0^{\frac{\pi}{3}}$

$= 2 \cdot \left[-2\cos x + \ln|\cos x|\right]_0^{\frac{\pi}{3}}$

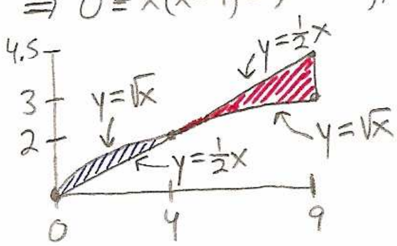
$= 2 \cdot \left[\left(-2\cos\frac{\pi}{3} + \ln\left|\cos\frac{\pi}{3}\right|\right) - \left(-2\cos 0 + \ln|\cos 0|\right)\right] = 2 \cdot \left[-2 \cdot \frac{1}{2} + \ln\frac{1}{2} - (-2 + \ln 1)\right]$

$= 2 \cdot \left[-1 + \ln\frac{1}{2} + 2 - 0\right] = 2 \cdot \left[1 + \ln\frac{1}{2}\right] = \boxed{2 + 2\ln\frac{1}{2}}$. Since $\ln\frac{1}{2} = \ln(2^{-1}) = -\ln 2$,

the answer is equivalent to $2 + 2(-\ln 2) = \boxed{2 - 2\ln 2}$.

17. $y = \sqrt{x}$, $y = \frac{1}{2}x$, $x = 9$. Curves intersect when $\sqrt{x} = \frac{1}{2}x \Rightarrow x = \frac{1}{4}x^2 \Rightarrow 4x = x^2 \Rightarrow 0 = x^2 - 4x$

$\Rightarrow 0 = x(x - 4) \Rightarrow x = 0, x = 4$.



$A = A_1 + A_2 = \int_0^4 \left[\sqrt{x} - \frac{1}{2}x\right] dx + \int_4^9 \left[\frac{1}{2}x - \sqrt{x}\right] dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}\frac{x^2}{2}\right]_0^4 + \left[\frac{1}{2}\frac{x^2}{2} - \frac{2}{3}x^{\frac{3}{2}}\right]_4^9$

$= \left[\frac{2}{3} \cdot 8 - 4\right] + \left[\left(\frac{81}{4} - 18\right) - \left(4 - \frac{2}{3} \cdot 8\right)\right]$

$= \frac{16}{3} - 4 + \frac{81}{4} - 18 - 4 + \frac{16}{3} = \frac{32}{3} - 26 + \frac{81}{4} = \frac{128 - 312 + 243}{12} = \boxed{\frac{59}{12}}$

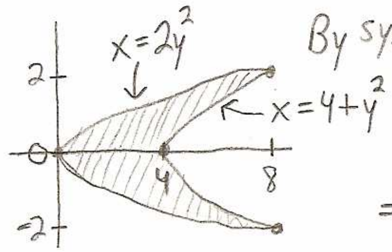
$26 \times 12 = 52 + 260 = 312$

$371 - 312 = 59$

19. $x = 2y^2$, $x = 4 + y^2$ Intersect when $2y^2 = 4 + y^2 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$.

x	y
8	-2
8	2
0	0

x	y
8	-2
8	2
4	0



By symmetry, $A = 2 \cdot \int_0^2 \left[4 + y^2 - 2y^2\right] dy$

$= 2 \cdot \int_0^2 \left[4 - y^2\right] dy$

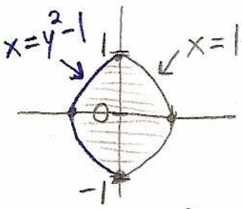
$= 2 \cdot \left[4y - \frac{y^3}{3}\right]_0^2 = 2 \cdot \left[8 - \frac{8}{3}\right]$

$= 2 \cdot \left[\frac{24 - 8}{3}\right] = \boxed{\frac{32}{3}}$

21. $x = 1 - y^2$, $x = y^2 - 1$. Intersect when $1 - y^2 = y^2 - 1 \Rightarrow 0 = 2y^2 - 2 = 0 = y^2 - 1 \Rightarrow y = \pm 1$.

x	y
0	-1
0	1
1	0

x	y
0	-1
0	1
-1	0



By symmetry, $A = 2 \cdot \int_0^1 \left[(1-y^2) - (y^2 - 1)\right] dy$

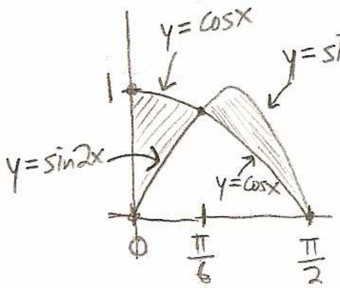
$$= 2 \cdot \int_0^1 \left[2 - 2y^2\right] dy$$

$$= 2 \cdot \left[2y - \tfrac{2}{3}y^3\right]_0^1 = 2 \cdot \left[2 - \tfrac{2}{3}\right] = 2 \cdot \tfrac{4}{3} = \boxed{\tfrac{8}{3}}$$

23. $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \frac{\pi}{2}$. Intersect when $\cos x = \sin 2x \Rightarrow \cos x = 2 \sin x \cos x$

$\Rightarrow 0 = 2 \sin x \cos x - \cos x \Rightarrow 0 = \cos x (2 \sin x - 1) \Rightarrow \cos x = 0$ OR $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$, etc. OR $x = \frac{\pi}{6}, \frac{5\pi}{6}$



$y = \cos x$

x	y
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0

$y = \sin 2x$

x	y
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0
$\frac{\pi}{4}$	1

$$A = \int_0^{\pi/6} \left[\cos x - \sin 2x\right] dx + \int_{\pi/6}^{\pi/2} \left[\sin 2x - \cos x\right] dx$$

$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

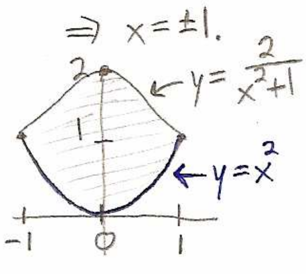
$$= \left[\left(\sin\tfrac{\pi}{6} + \tfrac{1}{2}\cos\tfrac{\pi}{3}\right) - \left(\sin 0 + \tfrac{1}{2}\cos 0\right)\right] + \left[\left(-\tfrac{1}{2}\cos\pi - \sin\tfrac{\pi}{2}\right) - \left(-\tfrac{1}{2}\cos\tfrac{\pi}{3} - \sin\tfrac{\pi}{6}\right)\right]$$

$$= \left(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2}\right) - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left[\left(-\tfrac{1}{2}\cdot(-1) - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right)\right]$$

$$= \cancel{\tfrac{1}{2}} + \tfrac{1}{4} - \cancel{\tfrac{1}{2}} + \left[\tfrac{1}{2} - 1 - \left(-\tfrac{1}{4} - \tfrac{1}{2}\right)\right] = \tfrac{1}{4} + \cancel{\tfrac{1}{2}} - \cancel{1} + \tfrac{1}{4} + \cancel{\tfrac{1}{2}} = \boxed{\tfrac{1}{2}}$$

25. $y = x^2$, $y = \frac{2}{x^2+1}$ Intersect when $x^2 = \frac{2}{x^2+1} \Rightarrow x^4 + x^2 - 2 = 0 \Rightarrow (x^2+2)(x^2-1) = 0$

$\Rightarrow x = \pm 1$. By symmetry, $A = 2 \cdot \int_0^1 \left[\frac{2}{x^2+1} - x^2\right] dx = 4 \int_0^1 \frac{1}{x^2+1} dx - 2 \int_0^1 x^2 \, dx$

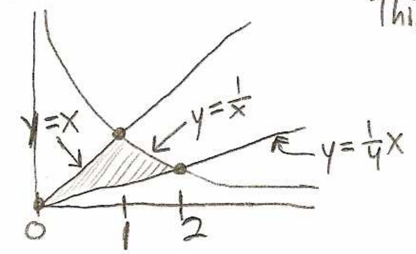


$$= 4\left[\tan^{-1} x\right]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1 = 4\left[\tan^{-1} 1 - \tan^{-1} 0\right] - \frac{2}{3}$$

$$= 4\left[\frac{\pi}{4} - 0\right] - \frac{2}{3} = \boxed{\pi - \frac{2}{3}}$$

27. $y = \frac{1}{x}$, $y = x$, $y = \frac{1}{4}x$, $x > 0$. First intersection pt. is $x = 0$. Second int. pt: $x = \frac{1}{x} \Rightarrow x = 1$.

Third int. pt.: $\frac{1}{x} = \frac{x}{4} \Rightarrow x^2 = 4 \Rightarrow x = 2$.



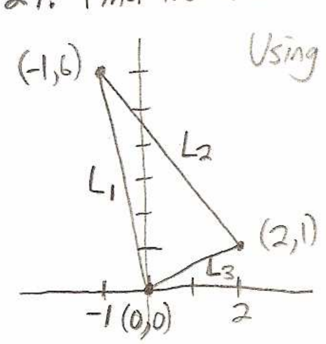
$$A = \int_0^1 \left[x - \tfrac{1}{4}x\right] dx + \int_1^2 \left[\tfrac{1}{x} - \tfrac{1}{4}x\right] dx$$

$$= \left[\frac{x^2}{2} - \frac{1}{4}\frac{x^2}{2}\right]_0^1 + \left[\ln|x| - \frac{1}{4}\frac{x^2}{2}\right]_1^2 = \text{(next page)}$$

27. $= \left[\frac{1}{2}-\frac{1}{8}\right] + \left[\left(\ln 2 - \frac{1}{2}\right) - \left(0-\frac{1}{8}\right)\right] = \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{8}} + \ln 2 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{8}} = \boxed{\ln 2}$

29. Find the area of the triangle with vertices at $(0,0)$, $(2,1)$, and $(-1,6)$.

L_1 is Line #1
L_2 is Line #2
L_3 is Line #3

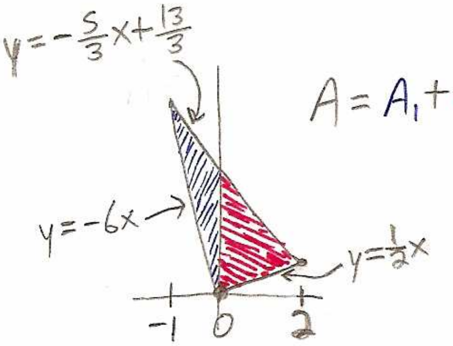


Using $y=mx+b$: For L_1, $m=-6$, $b=0$, so $\boxed{L_1 \text{ is } y=-6x.}$

For L_2: $m = \frac{1-6}{2-(-1)} = \frac{-5}{3}$, so, using $y=mx+b$ with $m=-\frac{5}{3}, x=2, y=1$:

$$1 = -\frac{5}{3}\cdot 2 + b$$
$$1 = -\frac{10}{3} + b \Rightarrow b = \frac{13}{3}.$$

So, $\boxed{L_2 \text{ is } y = -\frac{5}{3}x + \frac{13}{3}.}$ For L_3, $m=\frac{1}{2}$ and $b=0$, so $\boxed{L_3 \text{ is } y=\frac{1}{2}x}$



$$A = A_1 + A_2 = \int_{-1}^{0}\left[-\frac{5}{3}x+\frac{13}{3} - (-6x)\right]dx + \int_0^2\left[-\frac{5}{3}x + \frac{13}{3} - \frac{1}{2}x\right]dx$$

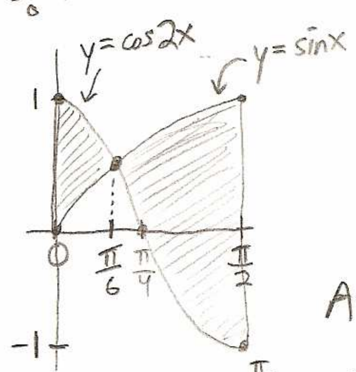
$$= \int_{-1}^{0}\left[-\frac{5}{3}x+\frac{13}{3}+\frac{18}{3}x\right]dx + \int_0^2\left[-\frac{10}{6}x+\frac{13}{3}-\frac{3}{6}x\right]dx$$

$$= \int_{-1}^{0}\left[\frac{13}{3}x+\frac{13}{3}\right]dx + \int_0^2\left[\frac{13}{3}-\frac{13}{6}x\right]dx$$

$$= \left[\frac{13}{3}\cdot\frac{x^2}{2}+\frac{13}{3}x\right]_{-1}^{0} + \left[\frac{13}{3}x - \frac{13}{6}\cdot\frac{x^2}{2}\right]_0^2 = \left[(0+0)-\left(\frac{13}{6}+\frac{-13}{3}\right)\right] + \left[\left(\frac{26}{3}-\frac{13}{3}\right)-(0-0)\right]$$

$$= -\frac{13}{6}+\frac{13}{3}+\frac{13}{3} = -\frac{13}{6}+\frac{26}{3} = -\frac{13}{6}+\frac{52}{6} = \frac{39}{6} = \boxed{\frac{13}{2}}$$

31. $\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x|\,dx$. Interpret the integral as the area of a region. Sketch the region.



The curves intersect when $\sin x = \cos 2x \Rightarrow \sin x = 1 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow (2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ OR $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\qquad x = \frac{3\pi}{2}$

$$A = A_1 + A_2 = \int_0^{\frac{\pi}{6}}\left[\cos 2x - \sin x\right]dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[\sin x - \cos 2x\right]dx$$

$$= \left[\frac{\sin 2x}{2}+\cos x\right]_0^{\frac{\pi}{6}} + \left[-\cos x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left[\left(\frac{1}{2}\sin\frac{\pi}{3}+\cos\frac{\pi}{6}\right)-(0+1)\right] + \left[(0-0)-\left(-\cos\frac{\pi}{6}-\frac{1}{2}\sin\frac{\pi}{3}\right)\right]$$

$$= \left[\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}-1\right] + \left[\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}\right] = \frac{2\sqrt{3}}{4}+\frac{2\sqrt{3}}{2}-1 = \frac{\sqrt{3}}{2}+\frac{2\sqrt{3}}{2}-1 = \boxed{\frac{3\sqrt{3}}{2}-1}$$