1. If $f(x)=\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence $R=10$, then $f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1}$ also has $R=10$. Theorem 2 states that power series representing the derivative and integral of $f(x)$ will have the same R as the power series for $f(x)$.

#3 - #9: Find a power series representation for the function and determine the interval of convergence.

The strategy is to use the fact that $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots$ (which converges for $|x|<1$)

3. $f(x)=\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=1}^{\infty}(-1)^n x^n$. Converges for $|-x|<1 \Rightarrow |x|<1$, so $I=(-1,1)$.

5. $f(x)=\frac{2}{3-x}=2\cdot\frac{1}{3(1-\frac{x}{3})}=\frac{2}{3}\cdot\frac{1}{1-\frac{x}{3}}=\underbrace{\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n}_{\text{Converges for } |\frac{x}{3}|<1 \Rightarrow |x|<3, \text{ so } I=(-3,3).}=2\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$.

7. $f(x)=\frac{x}{9+x^2}=x\cdot\frac{1}{9(1+\frac{x^2}{9})}=\frac{x}{9}\cdot\frac{1}{1-(-\frac{x^2}{9})}=\underbrace{\frac{x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n}=\frac{x}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}$

$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{9^{n+1}}$

Converges for $\left|-\frac{x^2}{9}\right|<1 \Rightarrow |x^2|<9 \Rightarrow |x|<3$, so $I=(-3,3)$.

9. $f(x)=\frac{1+x}{1-x}=\frac{1}{1-x}+\frac{x}{1-x}=\frac{1}{1-x}+x\cdot\frac{1}{1-x}=\underbrace{\sum_{n=0}^{\infty}x^n + x\cdot\sum_{n=0}^{\infty}x^n}_{\text{both converge for } |x|<1 \Rightarrow I=(-1,1).}=\sum_{n=0}^{\infty}x^n+\sum_{n=0}^{\infty}x^{n+1}$

$=(1+x+x^2+x^3+\cdots)+(x+x^2+x^3+\cdots)$

$=1+2x+2x^2+2x^3+\cdots = 1+2\sum_{n=0}^{\infty}x^{n+1}$ OR $1+2\sum_{n=1}^{\infty}x^n$.

alternate approach

9. $f(x)=\frac{1+x}{1-x}=\frac{1-x+2x}{1-x}=\frac{1-x}{1-x}+\frac{2x}{1-x}=1+2x\cdot\frac{1}{1-x}=1+2x\sum_{n=0}^{\infty}x^n$

$=1+2\sum_{n=0}^{\infty}x^{n+1}$ OR $1+2\sum_{n=1}^{\infty}x^n$. Converges for $|x|<1 \Rightarrow I=(-1,1)$.

a third approach

9. $f(x)=\frac{1+x}{1-x}=(1+x)\cdot\frac{1}{1-x}=(1+x)(1+x+x^2+x^3+\cdots)=(1+x+x^2+x^3+\cdots)+(x+x^2+x^3+\cdots)$

$=1+2x+2x^2+2x^3+\cdots=1+2\sum_{n=0}^{\infty}x^{n+1}$ OR $1+2\sum_{n=1}^{\infty}x^n$. Converges for $|x|<1 \Rightarrow I=(-1,1)$.

11. Express the function as the sum of a power series by first using partial fractions. Also find I.

$$f(x)=\frac{3}{x^2-x-2}=\frac{3}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}\Rightarrow 3=A(x+1)+B(x-2)$$
$$3=(A+B)x+(A-2B)\Rightarrow \begin{array}{l}A+B=0\\ A-2B=3\\ \hline 3B=-3\\ \boxed{B=-1}\\ \boxed{A=1}\end{array}$$

$$=\frac{1}{x-2}+\frac{-1}{x+1}=\frac{-1}{2-x}+\frac{-1}{1+x}=\frac{-1}{2(1-\frac{x}{2})}+\frac{-1}{1-(-x)}$$

$$=-\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}-\frac{1}{1-(-x)}=-\frac{1}{2}\underbrace{\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n}_{}-\underbrace{\sum_{n=0}^{\infty}(-x)^n}_{}=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}-\sum_{n=0}^{\infty}(-1)^nx^n=\text{next line}$$

Converges for $|\frac{x}{2}|<1\Rightarrow \boxed{|x|<2}$

Converges for $|-x|<1\Rightarrow \boxed{|x|<1}$

Must go with only the x values for which both series converge $\Rightarrow |x|<1$ and $I=(-1,1)$.

$$=-\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}+\sum_{n=0}^{\infty}(-1)^{n+1}x^n=\sum_{n=0}^{\infty}\left[(-1)^{n+1}-\frac{1}{2^{n+1}}\right]x^n.$$

13a. Use differentiation to find a power series representation for $f(x)=\frac{1}{(1+x)^2}$. What is R?

Let $g(x)=\frac{1}{1+x}=(1+x)^{-1}$. $g'(x)=-(1+x)^{-2}=-\frac{1}{(1+x)^2}=-f(x)$, so $g'(x)=-f(x)\Rightarrow f(x)=-g'(x)$.

Since $g(x)=\frac{1}{1-(-x)}=\underbrace{\sum_{n=0}^{\infty}(-x)^n}_{}=\sum_{n=0}^{\infty}(-1)^nx^n$, $g'(x)=\sum_{n=1}^{\infty}(-1)^n nx^{n-1}\Rightarrow f(x)=-\sum_{n=1}^{\infty}(-1)^n nx^{n-1}$

could have n−1 instead

Converges for $|-x|<1\Rightarrow|x|<1$, so $f(x)$ also converges for $|x|<1\Rightarrow R=1.$

Note: n starts at 1 now that we've taken the derivative.

$$=\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}.$$
$$=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n.$$

13b. Use part a to find a power series for $f(x)=\frac{1}{(1+x)^3}$.

$$\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right]=\frac{d}{dx}\left[(1+x)^{-2}\right]=-2(1+x)^{-3}=-2\cdot\frac{1}{(1+x)^3}=-2f(x),\text{ so } f(x)=-\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right]=$$

$$-\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1}=-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}(n+2)(n+1)x^n$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+2}(n+2)(n+1)x^n$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n.\quad R=1 \text{ from part a. by Theorem 2.}$$

13c. Use part b to find a power series for

$$f(x)=\frac{x^2}{(1+x)^3}=x^2\cdot\frac{1}{(1+x)^3}=x^2\cdot\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}\ \text{OR}\ \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)x^n.\quad R=1 \text{ from part b. by Thrm. 2.}$$

15. $f(x) = \ln(5-x)$ Find a power series for the function and determine R.

$$\frac{d}{dx}\left[\ln(5-x)\right] = \frac{1}{5-x}\cdot(-1) = -1\cdot\frac{1}{5(1-\frac{x}{5})} = -\frac{1}{5}\cdot\frac{1}{1-\frac{x}{5}} = -\frac{1}{5}\cdot\underbrace{\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n}_{} = -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}$$

Converges for $\left|\frac{x}{5}\right|<1 \Rightarrow |x|<5$, so $R=5$.

Now, $f(x) = \ln(5-x) = \int\left[-\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}\right]dx = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}}$.

If $x=0$, $\ln 5 = C$. So, $f(x) = \ln(5-x) = \ln 5 - \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}} = \ln 5 - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}$.

$R=5$ for $f(x)$ by Thrm. 2.

17. $f(x) = \frac{x^3}{(x-2)^2}$ Find a powerseries representation for the function and determine R.

$$\frac{d}{dx}\left[\frac{1}{2-x}\right] = \frac{d}{dx}\left[(2-x)^{-1}\right] = -1(2-x)^{-2}\cdot(-1) = \frac{1}{(2-x)^2} = \frac{1}{(x-2)^2}.$$

Converges for $\left|\frac{x}{2}\right|<1 \Rightarrow |x|<2$, so $R=2$.

$$\text{So } \frac{1}{(x-2)^2} = \frac{d}{dx}\left[\frac{1}{2-x}\right] = \frac{d}{dx}\left[\frac{1}{2(1-\frac{x}{2})}\right] = \frac{d}{dx}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}\right] = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{(n+1)x^n}{2^{n+2}}$$

Thus $f(x) = x^3\cdot\frac{1}{(x-2)^2} = x^3\sum_{n=0}^{\infty}\frac{(n+1)x^n}{2^{n+2}} = \sum_{n=0}^{\infty}\frac{(n+1)x^{n+3}}{2^{n+2}} = \sum_{n=3}^{\infty}\frac{(n-2)x^n}{2^{n-1}}$. $R=2$ by Thrm. 2.

23. Evaluate the indefinite integral as a powerseries. Find R.

$\int\frac{t}{1-t^8}dt$. $\quad \frac{t}{1-t^8} = t\cdot\frac{1}{1-t^8} = t\underbrace{\sum_{n=0}^{\infty}(t^8)^n}_{} = t\sum_{n=0}^{\infty}t^{8n} = \sum_{n=0}^{\infty}t^{8n+1}$

Converges for $|t^8|<1 \Rightarrow |t|<1$, so $R=1$.

So, $\int\frac{t}{1-t^8}dt = \int\left[\sum_{n=0}^{\infty}t^{8n+1}\right]dt = \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2} + C$.

$R=1$ by Thrm. 2.

25. Evaluate the indefinite integral as a power series. Find R.

$\int \frac{x-\tan^{-1}x}{x^3}\,dx.$ $\qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \underbrace{\sum_{n=0}^{\infty}(-x^2)^n}_{\text{Converges for } |-x^2|<1 \Rightarrow |x|<1 \Rightarrow R=1.} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

So, $\tan^{-1}x = \int\left[\sum_{n=0}^{\infty}(-1)^n x^{2n}\right]dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + \cancel{C}^{\,0}.$ (If $x=0$, $\tan^{-1}0 = 0 = C$, so $C=0$.)

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Now, $x - \tan^{-1}x = x - \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right)$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{2n+3}$$

And, $\frac{x-\tan^{-1}x}{x^3} = \frac{1}{x^3}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{2n+3} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n+3}$

Finally: $\int\frac{x-\tan^{-1}x}{x^3}\,dx = \int\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n+3}\right]dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)(2n+3)} + C.$ $\quad R=1$ by Thrm. 2.

OR $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{(2n-1)(2n+1)} + C = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{4n^2-1} + C$

27. Use a power series to approximate the definite integral to six decimal places ($|\text{error}| < 1\times10^{-6}$).

$\int_0^{0.2}\frac{1}{1+x^5}\,dx.$ $\qquad \frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \underbrace{\sum_{n=0}^{\infty}(-x^5)^n}_{\text{Converges for } |-x^5|<1 \Rightarrow |x|<1, \text{ so } R=1.} = \sum_{n=0}^{\infty}(-1)^n x^{5n}$, so (next line...)

$\int_0^{0.2}\frac{1}{1+x^5}\,dx = \int_0^{0.2}\left[\sum_{n=0}^{\infty}(-1)^n x^{5n}\right]dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{5n+1}}{5n+1}\Bigg]_0^{0.2} = x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \frac{x^{21}}{21} - \cdots\Bigg]_0^{0.2}$

$= .2 - \frac{.2^6}{6} + \underbrace{\frac{.2^{11}}{11}}_{\approx 2\times10^{-9} < 1\times10^{-6}} - \frac{.2^{16}}{16} + \frac{.2^{21}}{21} - \frac{.2^{26}}{26} + \cdots \approx .2 - \frac{.2^6}{6} \approx \boxed{.199989}$

$\approx 2\times10^{-9} < 1\times10^{-6}$, so by the Alternating Series Estimation Theorem (11.5), we only need the first two terms for our approximation to be accurate to 6 decimal places. (Continuing to add past the 2nd term will not change the 6th decimal place.)

Value of the 3rd term

29. Use a power series to approximate the definite integral to six decimal places,

$\int_0^{0.1} x \arctan(3x)\,dx.$ From #25, we know that $\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$

So, $\arctan(3x) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{2n+1}$

And, $x\arctan(3x) = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+2}}{2n+1}$

Finally:

$$\int_0^{0.1} x \arctan(3x)\,dx = \int_0^{0.1} \left[\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+2}}{2n+1} \right] dx = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+3}}{(2n+1)(2n+3)} \Bigg]_0^{0.1}$$

$$= \frac{\cancel{3}x^3}{1\cdot\cancel{3}} - \frac{3^3x^5}{3\cdot 5} + \frac{3^5x^7}{5\cdot 7} - \frac{3^7x^9}{7\cdot 9} + \frac{3^9x^{11}}{9\cdot 11} - \frac{3^{11}x^{13}}{11\cdot 13} + \cdots \Bigg]_0^{0.1}$$

$$= \underbrace{.1^3 - \frac{9}{5}(.1)^5 + \frac{3^5}{35}(.1)^7}_{\text{Need these three terms}} - \underbrace{\frac{3^7}{63}(.1)^9}_{(\approx 3\times 10^{-8} < 1\times 10^{-6}} + \frac{3^9}{99}(.1)^{11} - \frac{3^{11}}{(11\cdot 13)}(.1)^{13} + \cdots$$

adding past the 3rd term will not change the 6th decimal place)

$\approx .1^3 - \frac{9}{5}(.1)^5 + \frac{3^5}{35}(.1)^7$ (adding past the 3rd term will not change the 6th decimal place)

$\approx .000983$