11.8 homework Power series

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n$ OR $\sum_{n=0}^{\infty} c_n (x-a)^n$

2.b. The interval of convergence of a power series is the values of x for which the series converges. It is found by using the Ratio Test (or the Root Test if appropriate).

3. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| \to |x| < 1 \Leftrightarrow -1 < x < 1.$ $\boxed{R = 1}$ radius of convergence

If $x=-1$: $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges (Alternating Series Test).

If $x=1$: $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges (p-series, $p=\frac{1}{2}<1$).

Therefore $\boxed{\text{the interval of convergence is } [-1,1).}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{x^n}\right| \to |x| < 1 \Leftrightarrow -1 < x < 1.$ $\boxed{R=1}$

If $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)(-1)^{n-1}}{n^3} = \sum_{n=1}^{\infty} \frac{-1}{n^3} = -\sum_{n=1}^{\infty} \frac{1}{n^3}$,

which converges (constant multiple of a p-series, $p=3>1$).

If $x=1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$, which converges (Alternating Series Test).

Therefore $\boxed{I = [-1,1].}$

7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$ for all x.

Therefore the series converges for all values of x. $\boxed{R=\infty}$ and $\boxed{I = (-\infty, \infty)}$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2 x^n}\right| \to \frac{1}{2}|x| < 1 \Leftrightarrow |x| < 2$

$\Leftrightarrow -2 < x < 2.$

If $x=-2$: $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n n^2 (-2)^n}{2^n} = \sum_{n=1}^{\infty} n^2$, which diverges by the Test for Divergence.

If $x=2$: $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n^2$, which diverges by the Test for Divergence.

So, $\boxed{R=2}$ and $\boxed{I = (-2,2).}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1} x^{n+1}}{\sqrt[4]{n+1}} \cdot \frac{\sqrt[4]{n}}{2^n x^n}\right| \to 2|x|<1 \Leftrightarrow |x|<\frac{1}{2}$

$\Leftrightarrow -\frac{1}{2}<x<\frac{1}{2}.$

If $x=-\frac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n(-\frac{1}{2})^n}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$, which diverges ($p$-series, $p=\frac{1}{4}<1$).

If $x=\frac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n(\frac{1}{2})^n}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$, which converges (Alternating Series Test).

So $\boxed{R=\frac{1}{2}}$ and $\boxed{I=(-\frac{1}{2},\frac{1}{2}]}$.

13. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{4^{n+1}\ln(n+1)} \cdot \frac{4^n \ln n}{x^n}\right| \to \frac{1}{4}|x|<1 \Leftrightarrow |x|<4$

$\Leftrightarrow -4<x<4.$

If $x=-4$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(-4)^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$, which diverges by the Comparison Test with $\sum \frac{1}{n}$.

($\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$, and since $\sum \frac{1}{n}$ diverges, so must $\sum \frac{1}{\ln n}$)

If $x=4$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 4^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$, which converges by the Alternating Series Test.

So, $\boxed{R=4}$ and $\boxed{I=(-4,4]}$.

15. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n}\right| \to |x-2|<1 \Leftrightarrow -1<x-2<1$

$\Leftrightarrow 1<x<3.$

If $x=1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ converges (Alternating Series Test).

If $x=3$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by Comparison Test with the convergent p-series $\sum \frac{1}{n^2}$.

So, $\boxed{R=1}$ and $\boxed{I=[1,3]}$.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{3^n(x+4)^n}\right| \to 3|x+4|<1 \Leftrightarrow$

$|x+4|<\frac{1}{3} \Leftrightarrow$

$-\frac{1}{3}<x+4<\frac{1}{3} \Leftrightarrow$

$-\frac{13}{3} < x < -\frac{11}{3}$

If $x=-\frac{13}{3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(-\frac{13}{3}+\frac{12}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n(-\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges (Alt. Series Test).

If $x=-\frac{11}{3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(-\frac{11}{3}+\frac{12}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n(\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges ($p$-series, $p=\frac{1}{2}<1$).

So, $\boxed{R=\frac{1}{3}}$ and $\boxed{I=[-\frac{13}{3},-\frac{11}{3})}$.

19. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{x-2}{n}\right)^n$ Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x-2}{n}\right|^n} =$ (next line)

$= \lim_{n\to\infty} \frac{|x-2|}{n} = 0 < 1$ for all x. Therefore the series converges for all values of x.

$\boxed{R=\infty}$ and $\boxed{I=(-\infty,\infty)}$

OR

#19 using Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(x-2)^n}\right| = \left|\frac{(x-2)\cdot n^n}{(n+1)(n+1)^n}\right| \to \frac{|x-2|}{n+1} \to 0 < 1$ for all x.

Therefore the series converges for all values of x. $\boxed{R=\infty}$ $\boxed{I=(-\infty,\infty)}$

21. $\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n$, $b>0$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x-a)^{n+1}}{b^{n+1}} \cdot \frac{b^n}{n(x-a)^n}\right| \to$ (next line)

$\to \frac{|x-a|}{b} < 1 \Leftrightarrow |x-a| < b \Leftrightarrow -b < x-a < b \Leftrightarrow a-b < x < a+b.$

If $x = a-b$: $\sum_{n=1}^{\infty} \frac{n}{b^n}(a-b-a)^n = \sum_{n=1}^{\infty} \frac{n}{b^n}(-b)^n = \sum_{n=1}^{\infty}(-1)^n \cdot n$, which diverges by the Test for Divergence.

If $x = a+b$: $\sum_{n=1}^{\infty} \frac{n}{b^n} \cdot (a+b-a)^n = \sum_{n=1}^{\infty} \frac{n}{b^n}(b^n) = \sum_{n=1}^{\infty} n$, which diverges by the Test for Divergence.

So, $\boxed{R=b}$ and $\boxed{I=(a-b, a+b).}$

23. $\sum_{n=1}^{\infty} n!(2x-1)^n$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \left|\frac{(n+1)\cancel{n!}(2x-1)\cancel{(2x-1)^n}}{\cancel{n!}\cancel{(2x-1)^n}}\right| =$ next line

$= (n+1)|2x-1| \to \infty$ unless $|2x-1| = 0 \Leftrightarrow 2x-1=0 \Leftrightarrow 2x=1 \Leftrightarrow x=\frac{1}{2}.$

Therefore, the series is convergent only if $x=\frac{1}{2}$ and is divergent for all other x values.

$\boxed{R=0}$ $\boxed{I=\{\tfrac{1}{2}\}}$

25. $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(4x+1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x+1)^n}\right| \to |4x+1| < 1 \Leftrightarrow -1 < 4x+1 < 1$

$\Leftrightarrow -2 < 4x < 0$

$\Leftrightarrow -\frac{1}{2} < x < 0.$

If $x=-\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges (Alt. Series Test).

If $x=0$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2>1$). So $\boxed{R=\frac{1}{4}}$ and $\boxed{I=\left[-\frac{1}{2}, 0\right]}$

did #28 by accident! Oh well... $\rightarrow$ 28. $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| =$ (next line)

$$= \left| \frac{(n+1)!\,x^{n+1}}{1\cdot3\cdot5\cdot\dots\cdot(2[n+1]-1)} \cdot \frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{n!\,x^n} \right|$$

$$= \left| \frac{(n+1)\cdot\cancel{n!}\cdot x\cdot\cancel{x^n}}{1\cdot3\cdot5\cdot\dots\cdot(2n+1)} \cdot \frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{\cancel{n!}\,\cancel{x^n}} \right|$$

$$= \left| \frac{(n+1)\cdot x}{\cancel{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}\cdot(2n+1)} \cdot \frac{\cancel{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}}{1} \right|$$

$$= \left| \frac{(n+1)}{(2n+1)}\cdot x \right| \rightarrow \frac{1}{2}|x| < 1 \Leftrightarrow |x|<2 \Leftrightarrow -2<x<2.$$

If $x=\pm2$, $|a_n| = \dfrac{n!\,2^n}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)} = \dfrac{(1\cdot2\cdot3\cdot4\cdot\dots\cdot n)\overbrace{(2\cdot2\cdot2\cdot2\cdot\dots\cdot2)}^{n \text{ twos}}}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}$

$$= \frac{(1\cdot2)\cdot(2\cdot2)\cdot(3\cdot2)\cdot(4\cdot2)\cdot\dots\cdot(n\cdot2)}{1\cdot3\cdot5\cdot7\cdot\dots\cdot(2n-1)} \leftarrow \text{rearranging the factors in the numerator}$$

$= \dfrac{2\cdot4\cdot6\cdot8\cdot\dots\cdot2n}{1\cdot3\cdot5\cdot7\cdot\dots\cdot(2n-1)}$, and $a_n = \dfrac{2\cdot4\cdot6\cdot\dots\cdot2n}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)} > 1 \neq 0$, so the series diverges by the Test for Divergence.

Therefore $\boxed{R=2}$ and $\boxed{I=(-2,2)}$.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1\cdot3\cdot5\cdot7\cdot\dots\cdot(2n-1)}$ Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{x^{n+1}}{1\cdot3\cdot5\cdot\dots\cdot\underbrace{(2[n+1]-1)}_{2n+1}} \cdot \frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{x^n} \right|$

$$= \left| \frac{x\cdot\cancel{x^n}}{\cancel{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}\cdot(2n+1)} \cdot \frac{\cancel{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}}{\cancel{x^n}} \right| = \frac{|x|}{2n-1} \rightarrow 0<1 \text{ for all } x \text{ values.}$$

Therefore the series converges for all values of x. $\boxed{R=\infty}$ and $\boxed{I=(-\infty,\infty)}$.