<u>Ratio Test</u>: ~~used to test for absolute convergence~~

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ OR $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive (must use a different test to determine whether it is convergent or divergent).

Definition: $\sum a_n$ is <u>absolutely convergent</u> if $\sum |a_n|$ is convergent. Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is A.C.

Definition: $\sum a_n$ is <u>conditionally convergent</u> if it is convergent (usually by the Alternating Series Test) but not absolutely convergent. Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is C.C.

1. a. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, then $\sum a_n$ diverges by the Ratio Test.

   b. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, then $\sum a_n$ converges by the Ratio Test.

   c. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

3. $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$ Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n} = \lim_{n\to\infty} \frac{10\cdot\cancel{10^n}}{\cancel{10^n}}\cdot\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}$

$= \lim_{n\to\infty} \frac{10}{n+1} = 0 < 1 \Rightarrow$ the series is $\boxed{\text{absolutely convergent,}}$ and thus convergent.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ Notice that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{4}}}$ which diverges (p-series, $p = \frac{1}{4} < 1$), so the series is not absolutely convergent.

Since $\lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$ and $\left\{\frac{1}{\sqrt[4]{n}}\right\}$ is positive and decreasing, the original series is $\boxed{\text{conditionally convergent}}$ by the Alternating Series Test.

[The Ratio Test is inconclusive on #5, since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\sqrt[4]{n+1}}\cdot\frac{\sqrt[4]{n}}{1} = \sqrt[4]{\frac{n}{n+1}} = \sqrt[4]{\frac{1}{1+\frac{1}{n}}} \to 1$ as $n\to\infty$.]

7. $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k} = \frac{k+1}{k}\cdot\frac{2}{3} = \frac{1+\frac{1}{k}}{1}\cdot\frac{2}{3} \to \frac{2}{3} < 1$ (as $n\to\infty$)

Therefore, the series is $\boxed{\text{absolutely convergent}}$ and thus convergent by the Ratio Test.

9. $\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(1.1)^{n+1}}{(n+1)^4}\cdot\frac{n^4}{(1.1)^n} = 1.1\cdot\left(\frac{n}{n+1}\right)^4 = 1.1\cdot\left(\frac{1}{1+\frac{1}{n}}\right)^4$

$\to 1.1 > 1$ as $n\to\infty$. Therefore, the series is $\boxed{\text{divergent}}$ by the Ratio Test.

11. $\sum_{n=1}^{\infty} (-1)^n \frac{e^{\frac{1}{n}}}{n^3}$ Using the Comparison Test from 11.4: Note that for $n\geq 1$, $\frac{e^{\frac{1}{n}}}{n^3} \leq \frac{e}{n^3}$,

and $\sum_{n=1}^{\infty} \frac{e}{n^3} = e\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges since it is a constant multiple of a convergent p-series. ($p=3>1$)

So, by the Comparison Test, $\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^3}$ converges. Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{e^{\frac{1}{n}}}{n^3}$ is $\boxed{\text{absolutely convergent.}}$

[The Ratio Test is inconclusive on #11, since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{\frac{1}{n+1}}}{(n+1)^3}\cdot\frac{n^3}{e^{\frac{1}{n}}} = \left(\frac{n}{n+1}\right)^3\cdot\frac{e^{\frac{1}{n+1}}}{e^{\frac{1}{n}}} \to 1$ as $n\to\infty$.]

The Integral Test can also be used on #11:

$$\int_1^{\infty}\frac{e^{\frac{1}{x}}}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{\frac{1}{x}}}{x^3}\,dx \overset{\star}{=} \lim_{t\to\infty}\left[e^{\frac{1}{x}}\left(1-\frac{1}{x}\right)\right]_1^t = \lim_{t\to\infty}\left[e^{\frac{1}{t}}\left(1-\frac{1}{t}\right) - \cancel{e(1-1)}^{\,0}\right]$$

$$\star\int\frac{e^{\frac{1}{x}}}{x^3}\,dx = -\int e^{\frac{1}{x}}\cdot\frac{1}{x}\cdot\frac{-1}{x^2}\,dx = -\int ue^u\,du = -\left[ue^u - \int e^u\,du\right] = -ue^u + e^u$$

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}dx$

$U = u \quad V = e^u$

$dU = du \quad dV = e^u du$

$= e^u(1-u)$

$= e^{\frac{1}{x}}\left(1-\frac{1}{x}\right)$

$= e^0\cdot 1 = 1$. Since the integral converges, by the Integral Test the series $\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^3}$ also converges.

Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{e^{\frac{1}{n}}}{n^3}$ is $\boxed{\text{absolutely convergent}}$.

13. $\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{(n+2)4^{2(n+1)+1}}\cdot\frac{(n+1)4^{2n+1}}{10^n} =$ (next line)

$= 10\cdot\frac{n+1}{n+2}\cdot\frac{4^{2n+1}}{4^{2n+3}} = 10\cdot\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot\frac{\cancel{4^{2n+1}}}{4^2\cdot\cancel{4^{2n+1}}} \to \frac{10}{16} < 1$ as $n\to\infty$.

Therefore, by the Ratio Test the series is $\boxed{\text{absolutely convergent.}}$

15. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$ Using the Comparison Test from 11.4: $\frac{\arctan n}{n^2} \leq \frac{\frac{\pi}{2}}{n^2}$ for $n \geq 1$,

and $\sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{2}\right)}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it is a constant multiple of a convergent p-series). $[p=2>1]$

So, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$ converges.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$ is $\boxed{\text{absolutely convergent.}}$

17. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ Using the Comparison Test: $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$.

We know $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so by the Comparison Test $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ also must diverge,

which means that $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is not absolutely convergent.

Since $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ (and $\left\{\frac{1}{\ln n}\right\}_{n=2}^{\infty}$ is positive and decreasing), by the Alternating Series Test the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is $\boxed{\text{conditionally convergent.}}$

[The Ratio Test is inconclusive on #17: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{\ln(n+1)} \cdot \frac{\ln n}{1} =$ (next line)

$$= \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} = 1.$$]

19. $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n!}$ Since $\left|\cos \frac{n\pi}{3}\right| \leq 1$, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\left|\cos \frac{n\pi}{3}\right|}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{n!}$.

Using the Ratio Test on $\sum_{n=1}^{\infty} \frac{1}{n!}$: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$,

so $\sum_{n=1}^{\infty} \frac{1}{n!}$ is absolutely convergent.

Therefore, by the Comparison Test $\sum_{n=1}^{\infty} \frac{\left|\cos\left(\frac{n\pi}{3}\right)\right|}{n!}$ is convergent,

which implies that $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n!}$ is $\boxed{\text{absolutely convergent.}}$

The Root Test: useful when nth powers occur

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

21. $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1$, so the series is $\boxed{\text{absolutely convergent.}}$

23. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} = \sum_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^n$

Root Test: $\lim_{n\to\infty} \sqrt[\cancel{n}]{\left[\left(1+\frac{1}{n}\right)^n\right]^{\cancel{n}}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$, so the series is $\boxed{\text{divergent.}}$

Or think of it as:

$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1 \Rightarrow \boxed{\text{divergent}}$ by the Root Test.

25. $1 - \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} - \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots + (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!}$

The series is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!}$

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2(n+1)-1)}{(2(n+1)-1)!} \cdot \frac{(2n-1)!}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}$

$= \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n+1)}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} \cdot \frac{(2n-1)!}{(2n+1)!} = \frac{\cancel{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}(2n+1)}{\cancel{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}} \cdot \frac{\cancel{(2n-1)!}}{\cancel{(2n+1)}(2n)\cancel{(2n-1)!}}$

$= \frac{1}{2n} \to 0 < 1$ as $n\to\infty$. Therefore the series is $\boxed{\text{absolutely convergent}}$ by the Ratio Test.

27. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot\overbrace{(2(n+1))}^{2n+2}}{(n+1)!} \cdot \frac{n!}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}$

$= \frac{\cancel{2\cdot 4\cdot 6\cdots(2n)}(2n+2)}{\cancel{2\cdot 4\cdot 6\cdots(2n)}} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \frac{2n+2}{n+1} \to 2 > 1$ as $n \to \infty$. Therefore, the series is $\boxed{\text{divergent}}$ by the Ratio Test.

Or on #27, the Test for Divergence can be used, as follows:

$\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!} = \sum_{n=1}^{\infty} \frac{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot\cdots\cdot(2\cdot n)}{n!} = \sum_{n=1}^{\infty} \frac{2^n\cdot \cancel{n!}}{\cancel{n!}} = \sum_{n=1}^{\infty} 2^n$

Since $\lim_{n\to\infty} 2^n = \infty \neq 0$, the series is $\boxed{\text{divergent}}$ by the Test for Divergence.

Yet another approach is to note that $\sum_{n=1}^{\infty} 2^n$ is a divergent geometric series $(r = 2 > 1)$.

31. a. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ This is a convergent p-series $(p = 3 > 1)$, but the $\boxed{\text{Ratio Test is inconclusive:}}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)^3} \cdot \frac{n^3}{1} = \left(\frac{n}{n+1}\right)^3 = \left(\frac{1}{1+\frac{1}{n}}\right)^3 \to 1$ as $n \to \infty$. (No conclusion from Ratio Test.)

b. $\sum_{n=1}^{\infty} \frac{n}{2^n}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{n} \cdot \frac{\cancel{2^n}}{2\cdot\cancel{2^n}} \to \frac{1}{2} < 1$,

so the series is $\boxed{\text{convergent}}$ by the ratio test. (Conclusive)

c. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{(n+1)-1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{3^{n-1}} = \frac{3^n}{3^{n-1}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$

$= \frac{3\cdot\cancel{3^{n-1}}}{\cancel{3^{n-1}}} \cdot \sqrt{\frac{n}{n+1}} = 3\cdot\sqrt{\frac{1}{1+\frac{1}{n}}} \to 3 > 1$ as $n \to \infty$, so the series is $\boxed{\text{divergent}}$ by the Ratio Test. (conclusive)

d. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} = \sqrt{\frac{n+1}{n}} \cdot \frac{n^2+1}{n^2+2n+2} \to 1$,

so the $\boxed{\text{Ratio Test is inconclusive}}$. However, using the Comparison Test, we have

$\frac{\sqrt{n}}{1+n^2} \le \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges (p-series, $p = \frac{3}{2} > 1$), so

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ is convergent by the Comparison Test.