Alternating Series Test: If the Alternating Series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ (with $b_n>0$) satisfies:
(i) $b_{n+1} \le b_n$ ($\{b_n\}$ is decreasing) and (ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

Here is my suggestion for how to test an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$:

Take $\lim_{n\to\infty} b_n$. [Ingore the $(-1)^{n-1}$].

If $\lim_{n\to\infty} b_n = 0$, the series $\boxed{\text{converges}}$ provided that $\{b_n\}$ is positive and decreasing.

If $\lim_{n\to\infty} b_n \neq 0$, then $\lim_{n\to\infty}(-1)^{n-1}b_n$ Does Not Exist (D.N.E.), so the series $\boxed{\text{diverges}}$ by the Test for Divergence from 11.2.

1. a. An alternating series is a series whose terms are alternately positive and negative.
   b. An alt. series converges if (see box above).
   c. later

3. $\frac{4}{7} - \frac{4}{8} + \frac{4}{9} - \frac{4}{10} + \frac{4}{11} - \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n+6}$ $\left(b_n = \frac{4}{n+6}\right)$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{4}{n+6} = 0$ ✓, and $\{b_n\}$ is positive and decreasing*, so the series $\boxed{\text{converges}}$ by the Alternating Series Test.

[*Proof that $\{b_n\}$ is decreasing: $b_{n+1} = \frac{1}{(n+1)+6} = \frac{1}{n+7} < \frac{1}{n+6} = b_n$.]

5. $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{2n+1}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{2n+1} = 0$ ✓, and $\{b_n\}$ is positive and decreasing*, so the series is $\boxed{\text{Convergent}}$ by the Alternating Series Test.

[*Proof that $\{b_n\}$ is decreasing: $b_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3} < \frac{1}{2n+1} = b_n$.]

7. $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ $\quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$, so

$\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1}$ D.N.E. (does not exist)*. Therefore the series $\boxed{\text{diverges}}$ by the Test for Divergence.

$\ast$ For odd n: $\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1} = -\frac{3}{2}$ — different #s $\Rightarrow$ the limit DNE.

For even n: $\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1} = \frac{3}{2}$

<u>Test for Divergence</u> from 11.2 (reminder)

If $\lim_{n\to\infty} a_n$ D.N.E. or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

9. $\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{n}{10^n}}_{b_n}$ $\quad \lim_{n\to\infty} \frac{n}{10^n} = \frac{\infty}{\infty}$ Need L'Hospital's Rule. $\lim_{x\to\infty} \frac{x}{10^x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{10^x \cdot \ln 10} = 0$,

and $\{b_n\}$ is positive and decreasing*, so the series $\boxed{\text{converges}}$ by the Alternating Series Test.

*Proof $\{b_n\}$ is decreasing: Let $f(x) = \frac{x}{10^x}$. $f'(x) = \frac{10^x - x \cdot 10^x \ln 10}{(10^x)^2} = \frac{10^x(1 - x\ln 10)}{(10^x)^2}$

$= \frac{1 - x\ln 10}{10^x} < 0 \Leftrightarrow 1 - x\ln 10 < 0 \Leftrightarrow 1 < x\ln 10 \Leftrightarrow \frac{1}{\ln 10} < x \Leftrightarrow x > \frac{1}{\ln 10} \approx .43$,

so for $n \geq 1$, $\{b_n\}$ is decreasing.

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ $\quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = 0$, and $\{b_n\}$ is positive and decreasing*, so the series $\boxed{\text{converges}}$ by the Alternating Series Test.

*Proof $\{b_n\}$ is decreasing: Let $f(x) = \frac{x^2}{x^3+4}$. $f'(x) = \frac{(x^3+4)\cdot 2x - x^2 \cdot 3x^2}{(x^3+4)^2} = \frac{2x^4 + 8x - 3x^4}{(x^3+4)^2}$

$= \frac{8x - x^4}{(x^3+4)^2} < 0 \Leftrightarrow 8x - x^4 < 0 \Leftrightarrow x(8 - x^3) < 0 \Leftrightarrow x < 0$ or $x > 2$, so for $n > 2$, the series is decreasing.

$x = 0 \quad x = 2$

$- \;\; | \;\; + \;\; | \;\; -$

$\quad\;\; 0 \qquad\; 2$

13. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{\ln n} = \frac{\infty}{\infty}$ Need L'Hospital. $\lim_{n\to\infty} \frac{n}{\ln n} =$

$\lim_{x\to\infty} \frac{x}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} x = \infty \neq 0$, so $\lim_{n\to\infty} (-1)^n \frac{n}{\ln n}$ D.N.E. $\Rightarrow$

Therefore, the series $\boxed{\text{diverges}}$ by the Test for Divergence.

15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \frac{-1}{1^{3/4}} + \frac{1}{2^{3/4}} - \frac{1}{3^{3/4}} + \frac{1}{4^{3/4}} - \dots = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{3/4}}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, and $\{b_n\}$ is positive and decreasing, so the series $\boxed{\text{converges}}$ by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin 0 = 0$, and $\{b_n\}$ is positive for $n>1$ and decreasing for $n \geq 2$, so the series $\boxed{\text{converges}}$ by the Alternating Series Test.

19. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n \cdot n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n} \geq \lim_{n\to\infty} n = \infty$,

so $\lim_{n\to\infty} (-1)^n \frac{n^n}{n!}$ D.N.E. Therefore, the series $\boxed{\text{diverges}}$ by the Test for Divergence.

1c. The "remainder after n terms", R_n, is the error involved in using s_n (the sum of the first n terms) to approximate s (the sum of the series). The Alternating Series Estimation Theorem on p. 712 states that $|R_n| = |s - s_n| \leq b_{n+1}$, where b_{n+1} is the absolute value of the first neglected term.

23 and 25: Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

→ How to do these problems: Find the 1st term whose absolute value is less than the desired error. We need all terms before that one.

This is what the Alt. Series Estimation Theorem is trying to say.

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ $(|\text{error}| < 0.00005)$

$b_n = \frac{1}{n^6}$ $b_1 = \frac{1}{1}$ $b_2 = \frac{1}{2^6} \approx .02$ $b_3 = \frac{1}{3^6} = .001$ $b_4 = \frac{1}{4^6} \approx .0002$ $b_5 = \frac{1}{5^6} \approx .00006$

$b_6 = \frac{1}{6^6} \approx .00002 < 0.00005.$ We need the first 5 terms. to find the sum to the indicated accuracy.

[The series converges because $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n^6} = 0$ and $\{b_n\}$ is positive and decreasing.]

25. $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!}$ $(|\text{error}| < .000005)$

$b_n = \frac{1}{10^n n!}$ $\underbrace{b_0 = \frac{1}{1 \cdot 1} \quad b_1 = \frac{1}{10 \cdot 1} \quad b_2 = \frac{1}{10^2 \cdot 2} \quad b_3 = \frac{1}{10^3 \cdot 6},\ b_3 \approx .0002}_{\text{Need these terms (the 1st four terms)}}$ $b_4 = \frac{1}{10^4 \cdot 24}$ $b_4 \approx .000004 < .000005$

We need the first 4 terms to find the sum to the indicated accuracy.

[The series converges because $\lim_{n \to \infty} \frac{1}{10^n n!} = 0$ and $\{b_n\}$ is positive and decreasing.]

27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ Approximate the sum of the series correct to 4 decimal places. $(|\text{error}| < .0001)$

$b_n = \frac{1}{n^5}$ $b_1 = \frac{1}{1}$ $b_2 = \frac{1}{2^5}$ $b_3 = \frac{1}{3^5} \approx .004$ $b_4 = \frac{1}{4^5} \approx .00098$ $b_5 = \frac{1}{5^5} \approx .00032$

$b_6 = \frac{1}{6^5} \approx .00013$ $b_7 = \frac{1}{7^5} \approx .00006 < .0001$, so add the first 6 terms:

$S_6 = \frac{1}{1} - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} - \frac{1}{6^5} \approx .97208 \approx \boxed{.9721}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$ Approximate the sum of the series correct to 4 decimal places. $(|\text{error}| < .0001)$

$b_n = \frac{n^2}{10^n}$ $b_1 = \frac{1}{10}$ $b_2 = \frac{4}{100}$ $b_3 = \frac{9}{1000}$ $b_4 = \frac{16}{10{,}000} = .0016$ $b_5 = \frac{25}{100{,}000} = .00025$

$b_6 = \frac{36}{1{,}000{,}000} = .000036 < .0001$, so add the first 5 terms:

$S_5 = \frac{1}{10} - \frac{4}{100} + \frac{9}{1000} - \frac{16}{10^4} + \frac{25}{10^5} \approx .06765.$ Since the 7th term is negative, let's round

S_5 down to $\boxed{.0676}$.