11.4 homework Comparison Test and Limit Comparison Test

Comparison Test: Assuming $\sum a_n$ and $\sum b_n$ are series with positive terms:

① If $\sum b_n$ converges and $a_n \le b_n$ for all n, then $\sum a_n$ also converges.

② If $\sum b_n$ diverges and $a_n \ge b_n$ for all n, then $\sum a_n$ also diverges.

p.706 Note 1 clarifies that condition 1 or 2 need only hold for all $n \ge N$ where N is some fixed #.

1\. a. No conclusion can be drawn. This situation does not correspond to either part of the Comparison Test.

b. $\sum a_n$ converges by Part 1 of the Comparison Test.

2\. a. $\sum a_n$ diverges by Part 2 of the Comparison Test.

b. No conclusion can be drawn. This situation does not correspond to either part of the Comparison Test.

3\. $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ $\quad \frac{n}{2n^3+1} < \frac{n}{2n^3} = \frac{1}{2n^2} = \frac{1}{2}\cdot\frac{1}{n^2}$. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series, $p=2>1$), and $\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{1}{n^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges because constant multiples of convergent series are convergent. So, by the Comparison Test, the original series $\boxed{\text{converges}}$.

5\. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ $\quad \frac{n+1}{n\sqrt{n}} = \frac{n+1}{n^{\frac{3}{2}}} > \frac{n}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}}$. $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ diverges (p-series, $p=\frac{1}{2}\le 1$), so by the Comparison Test, $\sum_{n=1}^{\infty}\frac{n+1}{n\sqrt{n}}$ $\boxed{\text{diverges}}$.

7\. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ $\quad \frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$. $\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n$ converges (geometric series, $|r| = \frac{9}{10} < 1$), so by the Comparison Test, $\sum_{n=1}^{\infty}\frac{9^n}{3+10^n}$ $\boxed{\text{converges}}$.

9\. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ $\quad \frac{\cos^2 n}{n^2+1} \le \frac{1}{n^2+1} < \frac{1}{n^2}$. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series, $p=2>1$), so by the Comparison Test, the original series $\boxed{\text{converges}}$.

11\. $\sum_{n=1}^{\infty} \frac{n-1}{n\cdot 4^n}$ $\quad \frac{n-1}{n\cdot 4^n} < \frac{n}{n\cdot 4^n} = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$. $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ converges (geometric, $|r| = \frac{1}{4} < 1$), so by the Comparison Test, the original series $\boxed{\text{converges}}$.

positive for $n>1$

13. $\sum_{n=1}^{\infty} \dfrac{\arctan n}{n^{1.2}}$ $\quad \dfrac{\arctan n}{n^{1.2}} \le \dfrac{\frac{\pi}{2}}{n^{1.2}} = \dfrac{\pi}{2} \cdot \dfrac{1}{n^{1.2}}$ $\quad \sum_{n=1}^{\infty} \dfrac{1}{n^{1.2}}$ converges (p-series, $p = 1.2 > 1$),

positive for $n \ge 1$

and $\sum_{n=1}^{\infty} \dfrac{\pi}{2} \cdot \dfrac{1}{n^{1.2}} = \dfrac{\pi}{2} \sum_{n=1}^{\infty} \dfrac{1}{n^{1.2}}$ converges because a constant multiple of a convergent series is convergent. So, by the Comparison Test, $\sum_{n=1}^{\infty} \dfrac{\arctan n}{n^{1.2}}$ $\boxed{\text{converges}}$.

15. $\sum_{n=1}^{\infty} \dfrac{2+(-1)^n}{n\sqrt{n}}$ $\quad \dfrac{2+(-1)^n}{n\sqrt{n}} \le \dfrac{3}{n^{\frac{3}{2}}} = 3 \cdot \dfrac{1}{n^{\frac{3}{2}}}$. $\quad \sum_{n=1}^{\infty} \dfrac{1}{n^{\frac{3}{2}}}$ converges (p-series, $p = \frac{3}{2} > 1$),

positive for $n \ge 1$

and $\sum_{n=1}^{\infty} 3 \cdot \dfrac{1}{n^{\frac{3}{2}}} = 3 \sum_{n=1}^{\infty} \dfrac{1}{n^{\frac{3}{2}}}$ converges because it is a constant multiple of a convergent series. Therefore, by the Comparison Test, $\sum_{n \ge 1}^{\infty} \dfrac{2+(-1)^n}{n\sqrt{n}}$ $\boxed{\text{converges}}$.

<u>Limit Comparison Test</u>: Given $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$ where c is a finite # and $c > 0$, then either both series converge or both series diverge.

17. $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$ $\quad$ Let $a_n = \dfrac{1}{\sqrt{n^2+1}}$, $b_n = \dfrac{1}{n}$. $\quad \lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\left(\frac{1}{\sqrt{n^2+1}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}}$

$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n^2+1}}{\sqrt{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = \frac{1}{1} = 1 \leftarrow \text{a finite \#} > 0.$$

By the Limit Comparison Test, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$ also $\boxed{\text{diverges}}$.

Note: If we try to use the Comparison Test on 17, we'd hit a dead end: $\dfrac{1}{\sqrt{n^2+1}} < \dfrac{1}{\sqrt{n^2}} = \dfrac{1}{n}$. We would need our $\dfrac{1}{\sqrt{n^2+1}}$ to be $> \dfrac{1}{n}$ to use the Comparison Test.

19. $\sum_{n=1}^{\infty} \dfrac{1+4^n}{1+3^n}$ $\quad$ Let $a_n = \dfrac{1+4^n}{1+3^n}$, $b_n = \dfrac{4^n}{3^n} \left(= \left(\frac{4}{3}\right)^n\right)$ $\quad \lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{1+4^n}{1+3^n} \cdot \dfrac{3^n}{4^n} =$ next line

$$= \lim_{n\to\infty} \frac{1+4^n}{4^n} \cdot \frac{3^n}{1+3^n} = \lim_{n\to\infty} \frac{\left(\frac{1+4^n}{4^n}\right)}{\left(\frac{4^n}{4^n}\right)} \cdot \frac{\left(\frac{3^n}{3^n}\right)}{\left(\frac{1+3^n}{3^n}\right)} = \lim_{n\to\infty} \left(\cancel{\frac{1}{4^n}}^{\,0} + 1\right)\left(\frac{1}{\cancel{\frac{1}{3^n}}^{\,0} + 1}\right) = 1.$$

clever rearranging

a finite number > 0.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ diverges (geometric, $r = \frac{4}{3} > 1$), the original series also $\boxed{\text{diverges}}$ by the Limit Comparison Test.

19. (Alternate method using the Comparison Test)

$\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ $\quad \frac{1+4^n}{1+3^n} > \frac{1+4^n}{3^n+3^n} > \frac{4^n}{2\cdot 3^n} = \frac{1}{2}\cdot\left(\frac{4}{3}\right)^n.$ $\quad \sum_{n=1}^{\infty} \frac{1}{2}\cdot\left(\frac{4}{3}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n.$

Since $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$ is a divergent geometric series ($r=\frac{4}{3}>1$), $\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$ also diverges,

and therefore by the Comparison Test $\sum_{n=1}^{\infty}\frac{1+4^n}{1+3^n}$ $\boxed{\text{diverges}}$.

19. (Alternate method using the Test for Divergence from 11.2)

$\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ $\quad \lim_{n\to\infty}\frac{1+4^n}{1+3^n} = \lim_{n\to\infty}\frac{\left(\frac{1+4^n}{3^n}\right)}{\left(\frac{1+3^n}{3^n}\right)} = \lim_{n\to\infty}\frac{\overset{0}{\cancel{\frac{1}{3^n}}}+\left(\frac{4}{3}\right)^n}{\overset{0}{\cancel{\frac{1}{3^n}}}+1} = \lim_{n\to\infty}\left(\frac{4}{3}\right)^n = \infty \neq 0,$

so by the Test for Divergence the series $\boxed{\text{diverges}}$.

21. $\sum_{n=1}^{\infty}\frac{\sqrt{n+2}}{2n^2+n+1}$ Using the Limit Comparison Test with $a_n = \frac{\sqrt{n+2}}{2n^2+n+1}$ and $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n+2}}{2n^2+n+1}\cdot\frac{n^{\frac{3}{2}}}{1} \underset{\substack{\div \text{ top and} \\ \text{bot. by } n^2}}{=} \lim_{n\to\infty}\frac{\left(\frac{\sqrt{n+2}\cdot \cancel{n^{3/2}}}{\sqrt{n}\cdot \cancel{n^{3/2}}}\right)}{\left(\frac{2n^2+n+1}{n^2}\right)} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{2}{n}}}{2+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{2} \leftarrow \text{a finite \# > 0}$$

Therefore, since $\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}}_{\text{p-series, } p=\frac{3}{2}>1}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{2n^2+n+1}$ also $\boxed{\text{converges}}$.

23. $\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ Using the Limit Comparison Test with $a_n = \frac{2n+5}{(n^2+1)^2}$ and $b_n = \frac{1}{n^3}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n+5}{(n^2+1)^2}\cdot\frac{n^3}{1} = \lim_{n\to\infty}\frac{2n^4+5n^3}{n^4+2n^2+1} = \lim_{n\to\infty}\frac{\left(\frac{2n^4+5n^3}{n^4}\right)}{\left(\frac{n^4+2n^2+1}{n^4}\right)}$$

$$= \lim_{n\to\infty}\frac{2+\frac{5}{n}}{1+\frac{2}{n^2}+\frac{1}{n^4}} = \frac{2}{1} = 2 \leftarrow \text{a finite \# > 0.}$$

Therefore, since $\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^3}}_{\text{p-series, } p=3>1}$ converges, $\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ also $\boxed{\text{converges}}$.

25. $\sum_{n=1}^{\infty} \frac{n^2+n+1}{\sqrt{n^6+n^2+1}}$ Using the Limit Comparison Test with $a_n = \frac{n^2+n+1}{\sqrt{n^6+n^2+1}}$ and $b_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n+1}{\sqrt{n^6+n^2+1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\left(\frac{n^3+n^2+n}{n^3}\right)}{\frac{\sqrt{n^6+n^2+1}}{\sqrt{n^6}}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^4}+\frac{1}{n^6}}}$$

$\div$ num. and den. by n^3

$= \frac{1}{1} = 1$, which is a finite # > 0. Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{n^2+n+1}{\sqrt{n^6+n^2+1}}$ also $\boxed{\text{diverges}}$.

27. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n} = \sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^2}{e^n}$ Using the Limit Comparison Test with

$a_n = \frac{\left(1+\frac{1}{n}\right)^2}{e^n}$ and $b_n = \frac{1}{e^n} = \left(\frac{1}{e}\right)^n$: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{e^n} \cdot \frac{e^n}{1} =$ (next line)

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = (1+0)^2 = 1$, which is a finite # > 0.

Since $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ converges (geometric series, $|r| = \frac{1}{e} < 1$), $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$ also $\boxed{\text{converges}}$.

29. $\sum_{n=1}^{\infty} \frac{1}{n!}$ Notice $n! = \underbrace{n(n-1)(n-2)\cdots 2}_{n-1 \text{ of these}} \geq \underbrace{2\cdot 2\cdot 2\cdots 2}_{n-1 \text{ twos}} = 2^{n-1}$ for all $n \geq 2$.

We have $n! \geq 2^{n-1}$, so $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), so by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ also $\boxed{\text{converges}}$.

29. (Alternate method)

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \underset{n=1}{\frac{1}{1}} + \underset{n=2}{\frac{1}{2}} + \underset{n=3}{\frac{1}{6}} + \underset{n=4}{\frac{1}{24}} + \underset{n=5}{\frac{1}{120}} + \underset{n=6}{\frac{1}{720}} + \cdots$$

Note: For all $n \geq 4$, $\frac{1}{n!} < \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), $\sum_{n=1}^{\infty} \frac{1}{n!}$ also $\boxed{\text{converges}}$ by the Comparison Test.

(I used the information in Note1 on Page 706 to do #29)

31. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ Using the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \underbrace{\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}}_{\frac{0}{0}} \overset{H}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot\left(-\cancel{\frac{1}{n^2}}\right)}{\left(-\cancel{\frac{1}{n^2}}\right)} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1$$

a finite # > 0.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the Limit Comparison Test $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ must also $\boxed{\text{diverge}}$.