11.3 homework Integral Test; p-series

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{5}}}$ The function $f(x) = \frac{1}{\sqrt[5]{x}} = x^{-\frac{1}{5}}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^{\infty} x^{-\frac{1}{5}}\,dx = \lim_{t\to\infty} \int_1^t x^{-\frac{1}{5}}\,dx = \lim_{t\to\infty} \frac{5}{4}x^{\frac{4}{5}}\Big]_1^t = \frac{5}{4}\lim_{t\to\infty}\left[t^{\frac{4}{5}} - 1\right] = \infty.$ $\boxed{\text{Divergent}}$

5. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$ $f(x) = \frac{1}{(2x+1)^3}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t (2x+1)^{-3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t = -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2}\to 0 - \frac{1}{9}\right]$

$= -\frac{1}{4}\left[-\frac{1}{9}\right] = \frac{1}{36}.$ $\boxed{\text{Convergent}}$

$\frac{1}{2}\int (2x+1)^{-3}(2)\,dx = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\cdot\frac{u^{-2}}{-2} = -\frac{1}{4u^2} = -\frac{1}{4(2x+1)^2}$

$u = 2x+1$
$du = 2\,dx$

Note on #3: $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{5}}}$ is a p-series with $p = \frac{1}{5} < 1$, so it is $\boxed{\text{Divergent}}$.

If you see that it is a p-series, you don't need to do the Integral Test.

$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converges if } p>1 \text{ and diverges if } p\le 1}$ ← p-series

7. $\sum_{n=1}^{\infty} ne^{-n}$ $f(x) = xe^{-x}$ Check that it is decreasing:

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) = \frac{1-x}{e^x} < 0 \iff 1-x<0$
$\iff 1<x$
$\iff x>1.$

and it is decreasing on $(1,\infty)$,

The function is continuous and positive, and it is ultimately decreasing so it is o.k. to use the Integral Test. [Note: as long as the function is ultimately decreasing for all x greater than some #, you can use the integral test (it doesn't have to decrease on $[1,\infty)$, just on (c,∞) for $x>c$).]

$\int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}(x+1)\right]_1^t = -\lim_{t\to\infty}\left[\frac{t+1}{e^t} - \frac{2}{e}\right] = -\lim_{t\to\infty}\frac{t+1}{e^t} + \frac{2}{e}$

$\overset{H}{=} -\lim_{t\to\infty}\frac{1}{e^t} + \frac{2}{e}$

$= 0 + \frac{2}{e} = \frac{2}{e}.$ $\boxed{\text{Converges}}$

$u = x \quad v = -e^{-x}$
$du = dx \quad dv = e^{-x}dx$

$\int xe^{-x} = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1)$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{0.85}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ ← This is a p-series with $p = 0.85 < 1$, so it is divergent. A constant multiple of a divergent series is also divergent, so $\boxed{\text{Divergent}}$

11. $\displaystyle 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$ p-series, $p = 3 > 1 \Rightarrow \boxed{\text{Convergent}}$

13. $\displaystyle 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$ $\displaystyle f(x) = \frac{1}{2x-1}$ continuous, positive, and decreasing on $[1, \infty)$.

Integral Test: $\displaystyle\int_1^{\infty} \frac{1}{2x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x-1}\,dx = \lim_{t\to\infty} \left[\frac{1}{2}\ln(2x-1)\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\ln(2t-1) - 0\right]$

$= \infty.$ $\boxed{\text{Divergent}}$

$\displaystyle \frac{1}{2}\int \frac{1}{2x-1}\,dx(2) = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|2x-1|$

$u = 2x-1,\ du = 2\,dx$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3} = 5\sum_{n=1}^{\infty} \frac{1}{n^3} - 2\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$. By Theorem 11.2.8, constant multiples of convergent series also converge, and sums and differences of convergent series also converge. $\therefore \boxed{\text{Convergent}}$

Convergent p-series ($p = 3 > 1$)

Convergent p-series ($p = \frac{5}{2} > 1$)

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ $\displaystyle f(x) = \frac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty) \Rightarrow$ use Integral Test.

$\displaystyle\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right]$

$\displaystyle = \frac{1}{2}\left[\tan^{-1}\infty - \tan^{-1}\frac{1}{2}\right] = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\frac{1}{2}\right]$ = a constant. $\therefore \boxed{\text{Convergent}}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ $\displaystyle f(x) = \frac{\ln x}{x^3}$ $\displaystyle f'(x) = \frac{x^3\cdot\frac{1}{x} - \ln x\cdot 3x^2}{x^6} = \frac{x^2 - 3x^2\ln x}{x^6} = \frac{1 - 3\ln x}{x^4} < 0 \Leftrightarrow$

$1 - 3\ln x < 0 \Leftrightarrow 1 < 3\ln x \Leftrightarrow 3\ln x > 1 \Leftrightarrow \ln x > \frac{1}{3} \Leftrightarrow x > e^{\frac{1}{3}}$, so f is ultimately decreasing.

Since f is continuous, positive, and decreasing on $(e^{\frac{1}{3}}, \infty)$, the Integral Test applies.

$\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left[\frac{-\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^t = \lim_{t\to\infty}\left[\left(\frac{-\ln t}{2t^2} - \cancel{\frac{1}{4t^2}}^{\,0}\right) - \left(0 - \frac{1}{4}\right)\right]$ continued next page...

$u = \ln x \quad v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

$du = \frac{1}{x}\,dx \quad dv = x^{-3}\,dx$

$\displaystyle\Rightarrow \int \ln x \cdot x^{-3}\,dx = -\frac{\ln x}{2x^2} + \int \frac{1}{2}x^{-3}\,dx = -\frac{\ln x}{2x^2} + \frac{1}{2}\frac{x^{-2}}{-2} = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}$

19. continued from previous page

$$= \left[\lim_{t\to\infty} \frac{-\ln t}{2t^2}\right] + \frac{1}{4} \overset{H}{=} \left[\lim_{t\to\infty} \frac{-\frac{1}{t}}{4t}\right] + \frac{1}{4} = \left[\lim_{t\to\infty} -\frac{1}{4t^2}\right] + \frac{1}{4}$$

$= 0 + \frac{1}{4} = \frac{1}{4}$. The series is $\boxed{\text{convergent}}$.

21. $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ $f(x) = \frac{1}{x\ln x}$ is continuous, positive and decreasing on $[2,\infty)$, so the Integral Test applies.

$$\int_2^{\infty} \frac{1}{x\ln x}dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x}dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right]$$

$$= \ln\infty - \ln(\ln 2) = \infty. \boxed{\text{Divergent}}$$

$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln|u| = \ln|\ln x|$

$u = \ln x$, $du = \frac{1}{x}dx$

23. $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ $f(x) = \frac{e^{\frac{1}{x}}}{x^2}$ $f'(x) = \frac{\cancel{x^2}\cdot e^{\frac{1}{x}}\cdot\left(-\frac{1}{\cancel{x^2}}\right) - e^{\frac{1}{x}}\cdot 2x}{x^4} = \frac{-e^{\frac{1}{x}} - 2xe^{\frac{1}{x}}}{x^4}$

$= \frac{-e^{\frac{1}{x}}(1+2x)}{x^4}$. Notice the negative sign, so $f'(x) < 0$ whenever $1+2x > 0 \iff 2x > -1 \iff x > -\frac{1}{2}$.

Since $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$, the Integral Test applies.

$$\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{e^{\frac{1}{x}}}{x^2}dx = \lim_{t\to\infty}\left[-e^{\frac{1}{x}}\right]_1^t = -\lim_{t\to\infty}\left[e^{\frac{1}{t}} - e^1\right]$$

$$= -\left[e^0 - e\right] = -\left[1 - e\right] = e - 1.$$

$\boxed{\text{Convergent}}$

$-\int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int e^u du = -e^u = -e^{\frac{1}{x}}$

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}dx$

25. $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ $f(x) = \frac{1}{x^3+x}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x(x^2+1)}\,dx = \lim_{t\to\infty}\left[\ln x - \tfrac{1}{2}\ln(x^2+1)\right]_1^t = \lim_{t\to\infty}\left[\ln t - \tfrac{1}{2}\ln(t^2+1) - \left(0 - \tfrac{1}{2}\ln 2\right)\right]$$

$$= \left(\lim_{t\to\infty} \ln\left[\frac{t}{(t^2+1)^{\frac{1}{2}}}\right]\right) + \tfrac{1}{2}\ln 2$$

$$\int \frac{1}{x(x^2+1)}\,dx = \int\left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right)dx = \int\left[\frac{1}{x} - \frac{x}{x^2+1}\right]dx$$

$1 = A(x^2+1) + (Bx+C)x$

$1 = Ax^2 + A + Bx^2 + Cx$

$1 = (A+B)x^2 + Cx + A$

$\boxed{A=1}$ $A+B=0$ $\boxed{B=-1}$ $\boxed{C=0}$

$$= \ln|x| - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$= \ln(x) - \tfrac{1}{2}\ln(x^2+1)$$

$$\overset{H}{=} \lim_{t\to\infty} \frac{1}{\left[\frac{t}{(t^2+1)^{\frac{1}{2}}}\right]} \cdot \frac{(t^2+1)^{\frac{1}{2}} - t\cdot\frac{1}{2}(t^2+1)^{-\frac{1}{2}}\cdot 2t}{t^2+1} + \tfrac{1}{2}\ln 2$$

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$$= \lim_{t\to\infty} \frac{(t^2+1)^{\frac{1}{2}}}{t} \cdot \frac{\left[(t^2+1)^{\frac{1}{2}} - t^2(t^2+1)^{-\frac{1}{2}}\right]}{t^2+1} + \tfrac{1}{2}\ln 2$$

$$= \lim_{t\to\infty} \frac{\cancel{t^2}+1-\cancel{t^2}}{t(t^2+1)} + \tfrac{1}{2}\ln 2$$

$$= \lim_{t\to\infty} \frac{1}{t(t^2+1)} + \tfrac{1}{2}\ln 2$$

$$= 0 + \tfrac{1}{2}\ln 2$$

$$= \tfrac{1}{2}\ln 2. \quad \boxed{\text{Convergent}}$$

*A shorter way without having to use L'Hospital's Rule:

$$\left(\lim_{t\to\infty} \ln\left[\frac{t}{(t^2+1)^{\frac{1}{2}}}\right]\right) + \tfrac{1}{2}\ln 2$$

$$= \lim_{t\to\infty} \ln\left[\frac{t}{\sqrt{t^2+1}}\right] + \tfrac{1}{2}\ln 2$$

$$= \lim_{t\to\infty} \ln\left[\frac{\left(\frac{t}{t}\right)}{\frac{\sqrt{t^2+1}}{\sqrt{t^2}}}\right] + \tfrac{1}{2}\ln 2$$

$$= \lim_{t\to\infty} \ln\left[\frac{1}{\sqrt{1+\cancel{\frac{1}{t^2}}^{\,0}}}\right] + \tfrac{1}{2}\ln 2$$

$$= \ln 1 + \tfrac{1}{2}\ln 2 = 0 + \tfrac{1}{2}\ln 2 = \tfrac{1}{2}\ln 2. \quad \boxed{\text{Convergent}}$$