11.2 homework

1. a. A sequence is an ordered list of #s. A series is a sum of a list of #s.

 b. A convergent series is a series whose sequence of partial sums converges.
 A divergent series is a series whose sequence of partial sums diverges or for which $\lim_{n\to\infty} a_n \neq 0$.
 (A convergent series adds up to a #; a divergent series does not sum to a finite #.)

9. $a_n = \frac{2n}{3n+1}$.

a. Is the sequence $\{a_n\}$ convergent? $\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so yes, $\{a_n\}$ converges to $\frac{2}{3}$.

b. Is the series $\sum_{n=1}^{\infty} a_n$ convergent? No: since $\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \neq 0$, this is a divergent series by the "Test for Divergence" p. 692

For 11–19: Determine if the geometric series is convergent or divergent. If it converges, find its sum.

11. $3+2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots$ $a=3$, $r=\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, it $\boxed{\text{converges.}}$

$\text{Sum} = \frac{a}{1-r} = \frac{3}{1-\frac{2}{3}} = \frac{3}{\left(\frac{1}{3}\right)} = \boxed{9.}$

13. $3-4+\frac{16}{3}-\frac{64}{9}+\cdots$ $a=3$, $r=-\frac{4}{3}$. Since $|r| = \frac{4}{3} > 1$, it is $\boxed{\text{divergent}}$.

15. $\sum_{n=1}^{\infty} 6(0.9)^{n-1} = 6 + 6\cdot(0.9) + 6\cdot(0.9)^2 + \cdots$ $a=6$, $r=0.9$.

Since $|r| = 0.9 < 1$, it $\boxed{\text{converges.}}$ $\text{Sum} = \frac{a}{1-r} = \frac{6}{1-0.9} = \frac{6}{\left(\frac{1}{10}\right)} = \boxed{60.}$

17. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4\cdot 4^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{4}\cdot\left(-\frac{3}{4}\right)^{n-1} = \frac{1}{4} + \frac{1}{4}\cdot\left(-\frac{3}{4}\right) + \frac{1}{4}\cdot\left(-\frac{3}{4}\right)^2 + \cdots$

$a=\frac{1}{4}$, $r=-\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it $\boxed{\text{converges.}}$ $\text{Sum} = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\left(-\frac{3}{4}\right)} = \frac{\frac{1}{4}}{1+\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}} = \boxed{\frac{1}{7}}$

19. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{\pi^n}{3\cdot 3^n} = \sum_{n=0}^{\infty} \frac{1}{3}\cdot\left(\frac{\pi}{3}\right)^n = \frac{1}{3} + \frac{1}{3}\cdot\frac{\pi}{3} + \frac{1}{3}\cdot\left(\frac{\pi}{3}\right)^2 + \cdots$

$a=\frac{1}{3}$, $r=\frac{\pi}{3}$. Since $|r| = \frac{\pi}{3} \approx \frac{3.14}{3} > 1$, it $\boxed{\text{diverges}}$.

For 21–33: Determine whether the series converges or diverges. If it converges, find its sum.

21. $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$. The "harmonic series" $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (Example 7 proved it),

Short way (see next page for Long Way)

and a constant multiple of a divergent series is also a divergent series.

So, $\sum_{n=1}^{\infty} \frac{1}{2n}$ is $\boxed{\text{divergent}}$.

21. (Longer way)

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \cdots$$

$$S_n = \sum_{i=1}^{n} \frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \underbrace{\frac{1}{6} + \frac{1}{8}}_{=\frac{7}{24} > \frac{1}{4}} + \underbrace{\frac{1}{10} + \frac{1}{12} + \frac{1}{14}}_{\approx .25476 > \frac{1}{4}} + \underbrace{\frac{1}{16} + \frac{1}{18} + \frac{1}{20} + \frac{1}{22} + \frac{1}{24}}_{\approx .25518 > \frac{1}{4}} + \cdots + \frac{1}{2n}$$

S_n ↑ the nth partial sum

$$\lim_{n\to\infty} S_n > \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots$$ (the one-fourths continue to be added forever)

Therefore, $\lim_{n\to\infty} S_n = \infty$ which means that the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is $\boxed{\text{divergent.}}$

23. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$. $\lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \neq 0$, so the series $\boxed{\text{diverges}}$ by the Test for Divergence.

$$\lim_{k\to\infty} \frac{k^2}{k^2-1} = \lim_{k\to\infty} \frac{\left(\frac{k^2}{k^2}\right)}{\left(\frac{k^2-1}{k^2}\right)} = \lim_{k\to\infty} \frac{1}{1-\frac{1}{k^2}} = \frac{1}{1-0} = 1.$$

25. $$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n.$$ Two geometric series:

1st series — $a = \frac{1}{3}$, $r = \frac{1}{3}$. Since $|r|<1$ it converges. Sum $= \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}.$ → $\frac{1}{2} + 2 = \boxed{\frac{5}{2}}$

2nd series — $a = \frac{2}{3}$, $r = \frac{2}{3}$. Since $|r|<1$ it converges. Sum $= \frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$ $\boxed{\text{Convergent}}$

27. $\sum_{n=1}^{\infty} \sqrt[n]{2}$ $\lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \neq 0$, so the series $\boxed{\text{diverges}}$ by the Test for Divergence.

29. $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ $\lim_{n\to\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\frac{1}{2} \neq 0$, so the series $\boxed{\text{diverges}}$ by the Test for Divergence.

31. $\sum_{n=1}^{\infty} \arctan n$ $\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0 \Rightarrow$ the series $\boxed{\text{diverges}}$ by the Test for Divergence.

33. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right).$

1st series: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ is a geometric series with $a=\frac{1}{e}$, $r=\frac{1}{e}$

$$\text{Sum}=\frac{a}{1-r}=\frac{\frac{1}{e}\cdot e}{\left(1-\frac{1}{e}\right)\cdot e}=\boxed{\frac{1}{e-1}}.\ \text{(Converges)}$$

2nd series: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right).$

$1=A(n+1)+Bn$

$1=(A+B)n+A$

$\boxed{A=1}$ $A+B=0$ $\boxed{B=-1}$

$$S_n=\underbrace{\left(1-\cancel{\tfrac{1}{2}}\right)}_{n=1}+\underbrace{\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)}_{n=2}+\underbrace{\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)}_{n=3}+\cdots+\underbrace{\left(\cancel{\tfrac{1}{n-1}}-\cancel{\tfrac{1}{n}}\right)}_{n=n-1}+\underbrace{\left(\cancel{\tfrac{1}{n}}-\tfrac{1}{n+1}\right)}_{n=n}$$

the nth partial sum

This is a "collapsing" or "telescoping" sum (almost all terms cancel out).

So, $S_n=1-\frac{1}{n+1}$. $\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-0=\boxed{1}.$ (It converges.)

Thus, the original series is the sum of two convergent series and converges to $\boxed{\frac{1}{e-1}+1}=$

$$\frac{1}{e-1}+\frac{e-1}{e-1}=\boxed{\frac{e}{e-1}}.$$

35. $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n^2-1}=\sum_{n=2}^{\infty}\frac{2}{(n+1)(n-1)}=\sum_{n=2}^{\infty}\left(\frac{A}{n+1}+\frac{B}{n-1}\right)=\sum_{n=2}^{\infty}\left(\frac{-1}{n+1}+\frac{1}{n-1}\right)$

$2=A(n-1)+B(n+1)$

$2=(A+B)n+(-A+B)$

$A+B=0$

$-A+B=2$

$2B=2$ $\boxed{B=1}$ $\boxed{A=-1}$

$$S_n=\left(-\cancel{\tfrac{1}{3}}+1\right)+\left(-\cancel{\tfrac{1}{4}}+\tfrac{1}{2}\right)+\left(-\cancel{\tfrac{1}{5}}+\cancel{\tfrac{1}{3}}\right)+\left(-\cancel{\tfrac{1}{6}}+\cancel{\tfrac{1}{4}}\right)+\left(-\tfrac{1}{7}+\cancel{\tfrac{1}{5}}\right)$$ continued next line...

$$+\left(-\cancel{\tfrac{1}{8}}+\cancel{\tfrac{1}{6}}\right)+\cdots+\underbrace{\left(\cancel{\tfrac{-1}{n-1}}+\cancel{\tfrac{1}{n-3}}\right)}_{n-2}+\underbrace{\left(\tfrac{-1}{n}+\cancel{\tfrac{1}{n-2}}\right)}_{n-1}+\underbrace{\left(\tfrac{-1}{n+1}+\cancel{\tfrac{1}{n-1}}\right)}_{n}$$

$$S_n=1+\tfrac{1}{2}-\tfrac{1}{n}-\tfrac{1}{n+1}.$$

$\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right]=1+\frac{1}{2}-0-0=\frac{3}{2}.$ Thus, the series $\boxed{\text{converges}}$ to $\boxed{\frac{3}{2}}$.

37. $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n(n+3)}=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+3}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+3}\right)$

$$S_n=\left(1-\cancel{\tfrac{1}{4}}\right)+\left(\tfrac{1}{2}-\cancel{\tfrac{1}{5}}\right)+\left(\tfrac{1}{3}-\cancel{\tfrac{1}{6}}\right)+\left(\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{7}}\right)+\left(\cancel{\tfrac{1}{5}}-\cancel{\tfrac{1}{8}}\right)+\cdots+\underbrace{\left(\cancel{\tfrac{1}{n-4}}-\cancel{\tfrac{1}{n-1}}\right)}_{n-4}$$ continued next line

$3=A(n+3)+Bn$

$3=(A+B)n+3A$

$3A=3\Rightarrow\boxed{A=1}$

$A+B=0\Rightarrow\boxed{B=-1}$

$$+\underbrace{\left(\cancel{\tfrac{1}{n-3}}-\cancel{\tfrac{1}{n}}\right)}_{n-3}+\underbrace{\left(\cancel{\tfrac{1}{n-2}}-\tfrac{1}{n+1}\right)}_{n-2}+\underbrace{\left(\cancel{\tfrac{1}{n-1}}-\tfrac{1}{n+2}\right)}_{n-1}+\underbrace{\left(\cancel{\tfrac{1}{n}}-\tfrac{1}{n+3}\right)}_{n}$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right]=\frac{6+3+2}{6}=\frac{11}{6}.$$

Thus, the series $\boxed{\text{converges}}$ to $\boxed{\frac{11}{6}}$.

39. $\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$

$S_n = \left(e - \cancel{e^{\frac{1}{2}}}\right) + \left(\cancel{e^{\frac{1}{2}}} - \cancel{e^{\frac{1}{3}}}\right) + \left(\cancel{e^{\frac{1}{3}}} - \cancel{e^{\frac{1}{4}}}\right) + \cdots + \left(\cancel{e^{\frac{1}{n}}} - e^{\frac{1}{n+1}}\right) = e - e^{\frac{1}{n+1}}.$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(e - e^{\frac{1}{n+1}}\right) = e - e^0 = e - 1.$ The series $\boxed{\text{converges}}$ to $\boxed{e-1}$.

41. $0.\overline{2} = 0.2222\ldots = \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots$ geometric series; $a = \frac{2}{10}$; $r = \frac{1}{10}$

$\text{sum} = \frac{a}{1-r} = \frac{\frac{2}{10}}{1-\frac{1}{10}} = \frac{\frac{2}{\cancel{10}}}{\frac{9}{\cancel{10}}} = \boxed{\frac{2}{9}}$

43. $3.\overline{417} = 3.417417417\ldots = 3 + \underbrace{\frac{417}{1000} + \frac{417}{1{,}000{,}000} + \frac{417}{1{,}000{,}000{,}000} + \cdots}$

geometric series; $a = \frac{417}{1000}$, $r = \frac{1}{1000}$ $\text{sum} = \frac{a}{1-r} = \frac{\frac{417}{1000}}{1-\frac{1}{1000}} = \frac{\frac{417}{\cancel{1000}}}{\frac{999}{\cancel{1000}}} = \frac{417}{999}.$

Therefore, $3.\overline{417} = 3 + .\overline{417} = 3 + \frac{417}{999}$

$= \frac{2997}{999} + \frac{417}{999} = \frac{3414}{999} = \boxed{\frac{1138}{333}}$

45. $1.53\overline{42} = 1.53 + .00424242\ldots = 1.53 + \underbrace{\frac{42}{10{,}000} + \frac{42}{1{,}000{,}000} + \cdots}$ $a = \frac{42}{10{,}000}$, $r = \frac{1}{100}$

$\text{sum} = \frac{a}{1-r} = \frac{\frac{42}{10{,}000}}{1-\frac{1}{100}} = \frac{\frac{42}{\cancel{10{,}000}\,100}}{\frac{99}{\cancel{100}}} = \frac{42}{9900} = \frac{21}{4950} = \frac{7}{1650}$

Therefore, $1.53\overline{42} = 1.53 + .00\overline{42} = \frac{153}{100} + \frac{7}{1650} = \frac{5049+14}{3300} = \boxed{\frac{5063}{3300}}$

47. $\sum_{n=1}^{\infty}\frac{x^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$ converges for $\left|\frac{x}{3}\right| < 1 \Leftrightarrow -1 < \frac{x}{3} < 1 \Rightarrow \boxed{-3 < x < 3.}$

$a = \frac{x}{3}$, $r = \frac{x}{3} \Rightarrow \text{Sum} = \frac{a}{1-r} = \frac{\frac{x}{3}\cdot 3}{\left(1-\frac{x}{3}\right)\cdot 3} = \boxed{\frac{x}{3-x}}$ (Remember: geometric series converge if $|r| < 1$.)

49. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{x=0}^{\infty}(4x)^n$ this is a geometric series; $a = 1$, $r = 4x$. It converges if $|r| < 1 \Leftrightarrow$

$|4x| < 1 \Leftrightarrow -1 < 4x < 1 \Leftrightarrow \boxed{-\frac{1}{4} < x < \frac{1}{4}}$. $\text{Sum} = \frac{a}{1-r} = \boxed{\frac{1}{1-4x}}$

51. $\sum_{n=0}^{\infty}\frac{\cos^n x}{2^n} = \sum_{n=0}^{\infty}\left(\frac{\cos x}{2}\right)^n$ geometric series; $a = 1$, $r = \frac{\cos x}{2}$. It converges if $|r| < 1 \Leftrightarrow \left|\frac{\cos x}{2}\right| < 1$, which is true for all x. Thus the series $\boxed{\text{converges for all values of } x.}$

$\text{sum} = \frac{a}{1-r} = \frac{1 \cdot 2}{\left(1-\frac{\cos x}{2}\right)\cdot 2} = \boxed{\frac{2}{2-\cos x}}$