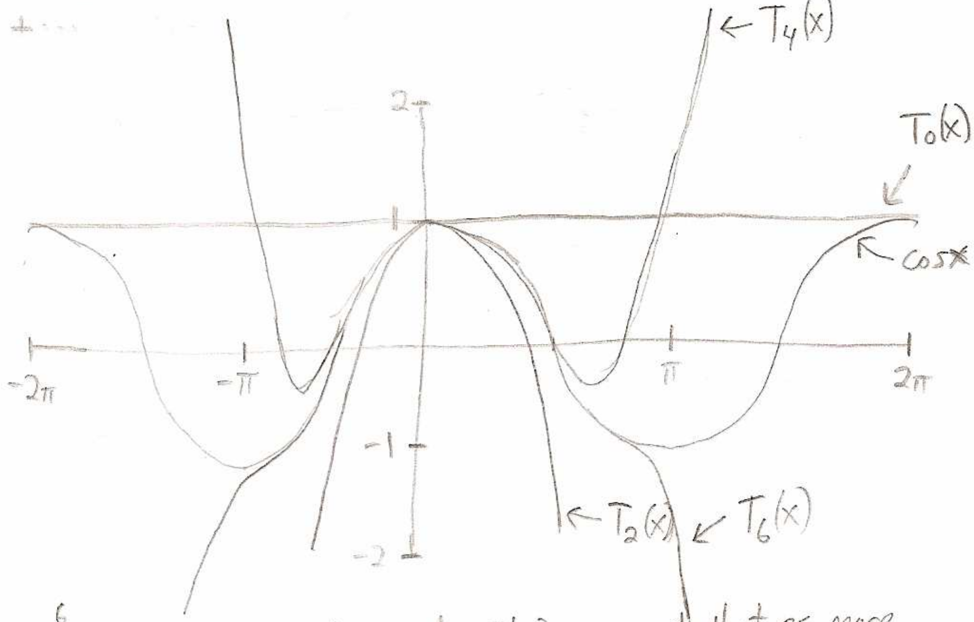
1. Find the Taylor polynomials up to degree 6 for $f(x) = \cos x$ centered at $a=0$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

degree	$T_n(x)$
0	$T_0(x) = 1$
1	$T_1(x) = 1$
2	$T_2(x) = 1 - \frac{x^2}{2!}$
3	$T_3(x) = 1 - \frac{x^2}{2!}$
4	$T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
5	$T_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
6	$T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$



From the graphs, it is apparent that as more terms are added the Taylor polynomials are a good approximation to $f(x) = \cos x$ on a larger and larger interval.

For #3 - #9: Find the Taylor polynomial $T_3(x)$ for the function f at the number a.

3. $f(x) = \frac{1}{x} = x^{-1}$, $a = 2$.

$f'(x) = -x^{-2} = \frac{-1}{x^2}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

$f'''(x) = -6x^{-4} = \frac{-6}{x^4}$

$f(2) = \frac{1}{2}$

$f'(2) = \frac{-1}{4}$

$f''(2) = \frac{2}{8} = \frac{1}{4}$

$f'''(2) = \frac{-6}{16} = -\frac{3}{8}$

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4 \cdot 2!}(x-2)^2 - \frac{3}{8 \cdot 3!}(x-2)^3$$

$$= \boxed{\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3}$$

5. $f(x) = \cos x$, $a = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = 0$

$f'(x) = -\sin x$ $\quad f'(\frac{\pi}{2}) = -1$

$f''(x) = -\cos x$ $\quad f''(\frac{\pi}{2}) = 0$

$f'''(x) = \sin x$ $\quad f'''(\frac{\pi}{2}) = 1$

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x-\tfrac{\pi}{2})^n$$

$$= f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})(x-\tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\tfrac{\pi}{2})^3$$

$$= 0 - 1(x-\tfrac{\pi}{2}) + 0 + \frac{1}{3!}(x-\tfrac{\pi}{2})^3$$

$$= \boxed{-(x-\tfrac{\pi}{2}) + \tfrac{1}{6}(x-\tfrac{\pi}{2})^3}$$

7. $f(x) = \arcsin x$, $a = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = x(1-x^2)^{-\frac{3}{2}} = \frac{x}{(1-x^2)^{\frac{3}{2}}}$

$f'''(x) = \frac{(1-x^2)^{\frac{3}{2}} - x\cdot\frac{3}{2}(1-x^2)^{\frac{1}{2}}(-\cancel{2}x)}{\left[(1-x^2)^{\frac{3}{2}}\right]^2} = \frac{(1-x^2)^{3/2} + 3x^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3} = \frac{(1-x^2)^{\frac{1}{2}}\left[(1-x^2)+3x^2\right]}{(1-x^2)^3}$

$= \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}}$

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 1$

$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + 1x + 0 + \frac{1}{3!}x^3$

$= \boxed{x + \frac{1}{6}x^3}$

9. $f(x) = xe^{-2x}$, $a = 0$

$f'(x) = e^{-2x} + x\cdot(-2e^{-2x}) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1-2x)$

$f''(x) = -2e^{-2x}(1-2x) + e^{-2x}(-2) = e^{-2x}(-2+4x-2) = e^{-2x}(4x-4)$

$f'''(x) = -2e^{-2x}(4x-4) + e^{-2x}\cdot 4 = e^{-2x}(-8x+8+4) = e^{-2x}(-8x+12)$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = -4$

$f'''(0) = 12$

$T_3(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + 1x - \frac{4}{2!}x^2 + \frac{12}{3!}x^3$

$= \boxed{x - 2x^2 + 2x^3}$

#13-21: Approximate f by a Taylor polynomial with degree n at the number a.

13. $f(x) = \sqrt{x}$, $a=4$, $n=2$, $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f''(4) = -\frac{1}{32}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \frac{-1}{4x^{3/2}}$

$T_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2$

$= \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$

b. Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_2(x)$ when x lies in the interval $[4, 4.2]$.

$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$, if $|f^{(n+1)}(x)| \le M$.

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} = \frac{3}{8x^{5/2}}$. Since f''' is decreasing on $[4, 4.2]$, $f'''(x) \le \frac{3}{8\cdot 4^{\frac{5}{2}}} = \frac{3}{256} = M$.

also, $|x-a| \le 0.2$, so $|R_2(x)| \le \frac{\left(\frac{3}{256}\right)}{3!}(0.2)^3 \approx .000015625$. Therefore, the error is $\le .000015625$.

c. From the graph of $|R_2(x)| = |f(x) - T_2(x)| = \left|\sqrt{x} - (2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2\right|$, the error $\le .000015153$.

(see next page below 15c)

15. $f(x) = x^{\frac{2}{3}}$, $a=1$, $n=3$, $0.8 \le x \le 1.2$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{1/3}}$ $\quad f(1)=1,\ f'(1)=\frac{2}{3},\ f''(1) = -\frac{2}{9},\ f'''(1) = \frac{8}{27}$

$f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} = \frac{-2}{9x^{4/3}}$ $\quad T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)}{3!}(x-1)^3$

$f'''(x) = \frac{8}{27}x^{-\frac{7}{3}} = \frac{8}{27x^{7/3}}$ $\quad = 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{\cancel{8}4}{27\cdot 3\cdot \cancel{2}}(x-1)^3$

$$= \boxed{1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3}$$

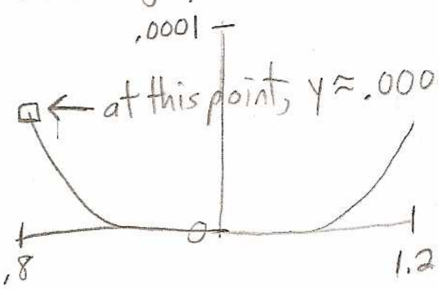
b. $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$, if $|f^{(n+1)}(x)| \le M$.

$f^{(4)}(x) = -\frac{7}{3}\cdot\frac{8}{27}x^{\frac{-10}{3}} = \frac{-56}{81x^{10/3}}$. $\quad |f^{(4)}(x)| = \underbrace{\frac{56}{81x^{10/3}}}_{\text{This fraction is maximum on } [.8,1.2] \text{ when } x=.8} \le \frac{56}{81\cdot(0.8)^{10/3}} \approx 1.45457589 = M.$

also, $|x-1| \le 0.2$.

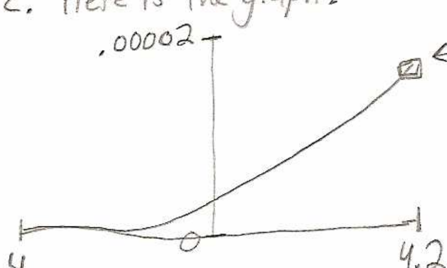
So, $|R_3(x)| \le \frac{1.45457589}{4!}(0.2)^4 \approx .0000969717260\text{2}$. Thus, the error $\le .00009697172602$.

c. From the graph of $|R_3(x)| = |f(x) - T_3(x)| = \left|x^{\frac{2}{3}} - \left(1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3\right)\right|$:



$\leftarrow$ at this point, $y \approx .000053284$, so the error $\le .000053284$ on the interval $[0.8, 1.2]$.

13c. Here is the graph:



(Use Trace on calculator)

$\leftarrow$ at this point, $y \approx .000015153$, so it appears from the graph that the error $\le .000015153$.

23. Use the result from Exercise 5 to estimate $\cos 80^\circ$ correct to five decimal places.

From #5, the 3rd degree Taylor polynomial for $f(x) = \cos x$ at $\frac{\pi}{2}$ is: $T_3(x) = -(x-\frac{\pi}{2}) + \frac{1}{6}(x-\frac{\pi}{2})^3$.

$80^\circ \cdot \frac{\pi}{180^\circ} = \frac{4\pi}{9}$, so $\cos \frac{4\pi}{9} \approx T_3\left(\frac{4\pi}{9}\right) \approx -\left(\frac{4\pi}{9} - \frac{\pi}{2}\right) + \frac{1}{6}\left(\frac{4\pi}{9} - \frac{\pi}{2}\right)^3$

$\approx .17365$