11.10 homework Taylor and Maclaurin Series

- Taylor Series for the function f at a:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\cdots$$

- Maclaurin Series for f (a=0):

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots$$

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5. Find the Maclaurin series for f(x) using the definition of a Maclaurin series. Also find the radius of convergence.

$$f(x)=(1-x)^{-2}=\frac{1}{(1-x)^2}$$
$$f'(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}=\frac{2}{(1-x)^3}$$
$$f''(x)=-3\cdot2(1-x)^{-4}(-1)=3\cdot2(1-x)^{-4}=\frac{3\cdot2}{(1-x)^4}$$
$$f'''(x)=-4\cdot3\cdot2(1-x)^{-5}(-1)=4\cdot3\cdot2(1-x)^{-5}=\frac{4\cdot3\cdot2}{(1-x)^5}$$

$f(0)=1$

$f'(0)=2$

$f''(0)=3\cdot2$

$f'''(0)=4\cdot3\cdot2$

$f^{(4)}(0)=5\cdot4\cdot3\cdot2$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots$$

$$=1+2x+\frac{3.2}{2!}x^2+\frac{4\cdot3\cdot2}{3!}x^3+\frac{5\cdot4\cdot3\cdot2}{4!}x^4+\cdots$$

$$=\underset{n=0}{1}+\underset{n=1}{2x}+\underset{n=2}{3x^2}+\underset{n=3}{4x^3}+\underset{n=4}{5x^4}+\underset{n=5}{6x^5}+\underset{n=6}{7x^6}+\cdots = \boxed{\sum_{n=0}^{\infty}(n+1)x^n}$$

Ratio Test to find radius of convergence:
$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|\to|x|<1\Rightarrow\boxed{R=1.}$$

7. Find Maclaurin series for f(x) = sin πx using the definition of a Maclaurin series. Also find R.

$f(x)=\sin\pi x$ | $f(0)=0$

$f'(x)=\pi\cos\pi x$ | $f'(0)=\pi$

$f''(x)=-\pi^2\sin\pi x$ | $f''(0)=0$

$f'''(x)=-\pi^3\cos\pi x$ | $f'''(0)=-\pi^3$

$f^{(4)}(x)=\pi^4\sin\pi x$ | $f^{(4)}(0)=0$

$f^{(5)}(x)=\pi^5\cos\pi x$ | $f^{(5)}(0)=\pi^5$

$f^{(6)}(x)=-\pi^6\sin\pi x$ | $f^{(6)}(0)=0$

$f^{(7)}(x)=-\pi^7\cos\pi x$ | $f^{(7)}(0)=-\pi^7$

$$f(x)=\sin\pi x=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\cancel{f(0)}^{0}+f'(0)x+\frac{\cancel{f''(0)}^{0}}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

$$=\underset{n=0}{\pi x}-\underset{n=1}{\frac{\pi^3}{3!}x^3}+\underset{n=2}{\frac{\pi^5}{5!}x^5}-\underset{n=3}{\frac{\pi^7}{7!}x^7}+\underset{n=4}{\frac{\pi^9}{9!}x^9}-\underset{n=5}{\frac{\pi^{11}}{11!}x^{11}}+\cdots$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}x^{2n+1}}$$

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7. continued

Ratio Test to find R: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\pi^{2(n+1)+1}}{(2(n+1)+1)!} x^{2(n+1)+1} \cdot \frac{(2n+1)!}{\pi^{2n+1} x^{2n+1}}\right|$

$= \left|\frac{\pi^{2n+3} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\pi^{2n+1} x^{2n+1}}\right| = \left|\frac{\pi^2 x^2}{(2n+3)(2n+2)}\right| \to 0 < 1$ for all x, so $\boxed{R = \infty.}$

9. Find the Maclaurin series for $f(x) = e^{5x}$ using the definition of a Maclaurin series. Also find R.

$f(x) = e^{5x}$	$f(0) = 1$
$f'(x) = 5e^{5x}$	$f'(0) = 5$
$f''(x) = 5^2 e^{5x}$	$f''(0) = 5^2$
$f'''(x) = 5^3 e^{5x}$	$f'''(0) = 5^3$
$f^{(4)}(x) = 5^4 e^{5x}$	$f^{(4)}(0) = 5^4$

$f(x) = e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$= \underset{n=0}{1} + \underset{n=1}{5x} + \underset{n=2}{\frac{5^2}{2!}x^2} + \underset{n=3}{\frac{5^3}{3!}x^3} + \underset{n=4}{\frac{5^4}{4!}x^4} + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n}$

Ratio Test to find R: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n x^n}\right| = \left|\frac{5x}{n+1}\right| \to 0 < 1$ for all x, so $\boxed{R = \infty.}$

13. Find the Taylor series for f(x) centered at the given value of a. $f(x) = x^4 - 3x^2 + 1$, $a = 1$.

$f(x) = x^4 - 3x^2 + 1$	$f(1) = -1$
$f'(x) = 4x^3 - 6x$	$f'(1) = -2$
$f''(x) = 12x^2 - 6$	$f''(1) = 6$
$f'''(x) = 24x$	$f'''(1) = 24$
$f^{(4)}(x) = 24$	$f^{(4)}(1) = 24$
$f^{(5)}(x) = 0$	$f^{(5)}(1) = 0$
$f^{(n)}(x) = 0$ for $n \geq 5$	

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(1)}{3!}(x-a)^3 + \cdots$

$= -1 - 2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$ ← This is the last term of the series since $f^{(n)}(1) = 0$ for all terms past this term.

$= \boxed{-1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4}$

Any finite series converges for all x, so $\boxed{R = \infty.}$

15. Find the Taylor series for $f(x) = e^x$ centered at $a = 3$.

$f(x) = e^x$	$f(3) = e^3$
$f'(x) = e^x$	$f'(3) = e^3$
$f^{(n)}(x) = e^x$	$f^{(n)}(3) = e^3$

for all n

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \cdots$

$= \underset{n=0}{e^3} + \underset{n=1}{e^3(x-3)} + \underset{n=2}{\frac{e^3}{2!}(x-3)^2} + \underset{n=3}{\frac{e^3}{3!}(x-3)^3} + \underset{n=4}{\frac{e^3}{4!}(x-4)^4} + \cdots$

$= \boxed{\sum_{n=0}^{\infty} \frac{e^3(x-3)^n}{n!}}$

Ratio Test to find R: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{e^3}(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{\cancel{e^3}(x-3)^n}\right| = \left|\frac{x-3}{n+1}\right| \to 0 < 1$ for all x, so $\boxed{R = \infty.}$

17. Find the Taylor series for $f(x) = \cos x$ centered at $a = \pi$.

$f(x) = \cos x \quad f(\pi) = -1$
$f'(x) = -\sin x \quad f'(\pi) = 0$
$f''(x) = -\cos x \quad f''(\pi) = 1$
$f'''(x) = \sin x \quad f'''(\pi) = 0$
$f^{(4)}(x) = \cos x \quad f^{(4)}(\pi) = -1$
$f^{(5)}(x) = -\sin x \quad f^{(5)}(\pi) = 0$
$f^{(6)}(x) = -\cos x \quad f^{(6)}(\pi) = 1$
$f^{(7)}(x) = \sin x \quad f^{(7)}(\pi) = 0$
$f^{(8)}(x) = \cos x \quad f^{(8)}(\pi) = -1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n = f(\pi) + \cancel{f'(\pi)(x-\pi)}^{\,0} + \frac{f''(\pi)}{2!}(x-\pi)^2 + \cancel{\frac{f'''(\pi)}{3!}(x-\pi)^3}^{\,0} + \cdots$$

$$= \underset{n=0}{-1} + \underbrace{\frac{(x-\pi)^2}{2!}}_{n=1} - \underbrace{\frac{(x-\pi)^4}{4!}}_{n=2} + \underbrace{\frac{(x-\pi)^6}{6!}}_{n=3} - \underbrace{\frac{(x-\pi)^8}{8!}}_{n=4} + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-\pi)^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(x-\pi)^{2n}}\right|$$

$$= \left|\frac{(x-\pi)^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(x-\pi)^{2n}}\right| = \left|\frac{(x-\pi)^2}{(2n+2)(2n+1)}\right| \to 0 < 1 \ \forall x, \text{ so } \boxed{R = \infty}.$$

19. Find the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $a = 9$.

$f(x) = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} \qquad f(9) = \frac{1}{3}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = \frac{-1}{2x^{3/2}} \qquad f'(9) = \frac{-1}{2\cdot 3^3}$

$f''(x) = \frac{3}{2}\cdot\frac{1}{2}x^{-\frac{5}{2}} = \frac{3\cdot 1}{2^2 x^{5/2}} \qquad f''(9) = \frac{3\cdot 1}{2^2\cdot 3^5}$

$f'''(x) = -\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}x^{-\frac{7}{2}} = \frac{-5\cdot 3\cdot 1}{2^3 x^{7/2}} \qquad f'''(9) = \frac{-5\cdot 3\cdot 1}{2^3\cdot 3^7}$

$f^{(4)}(x) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}x^{-\frac{9}{2}} = \frac{7\cdot 5\cdot 3\cdot 1}{2^4 x^{9/2}} \qquad f^{(4)}(9) = \frac{7\cdot 5\cdot 3\cdot 1}{2^4\cdot 3^9}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(9)}{n!}(x-9)^n = f(9) + f'(9)(x-9) + \frac{f''(9)}{2!}(x-9)^2 + \frac{f'''(9)}{3!}(x-9)^3 + \frac{f^{(4)}(9)}{4!}(x-9)^4 + \cdots$$

$$= \underset{n=0}{\frac{1}{3}} - \underset{n=1}{\frac{1}{2\cdot 3^3}(x-9)} + \underset{n=2}{\frac{1\cdot 3}{2^2\cdot 3^5\cdot 2!}(x-9)^2} - \underset{n=3}{\frac{1\cdot 3\cdot 5}{2^3\cdot 3^7\cdot 3!}(x-9)^3} + \underset{n=4}{\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 3^9\cdot 4!}(x-9)^4} + \cdots$$

$$= \boxed{\frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1\cdot 3\cdot 5\cdot 7\cdot \cdots \cdot (2n-1)}{2^n 3^{2n+1}\cdot n!}(x-9)^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2(n+1)-1)\cdot (x-9)^{n+1}}{2^{n+1}3^{2(n+1)+1}(n+1)!} \cdot \frac{2^n 3^{2n+1} n!}{1\cdot 3\cdot 5\cdots(2n-1)(x-9)^n}\right|$$

$$= \left|\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n+1)}{1\cdot 3\cdot 5\cdots(2n-1)} \cdot \frac{1}{2} \cdot \frac{3^{2n+1}}{3^{2n+3}} \cdot \frac{(x-9)}{n+1}\right| = \left|\frac{(2n+1)\cdot(x-9)}{2\cdot 3^2\cdot(n+1)}\right| \to \frac{1}{9}|x-9| < 1$$

$$\Rightarrow |x-9| < 9 \Rightarrow \text{next line}$$

$$-9 < x-9 < 9 \Rightarrow 0 < x < 18. \ \boxed{R = 9}.$$

#29-37: Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (R=\infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad (R=\infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (R=\infty)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad (R=1)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \quad (R=1)$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots \quad (R=1)$$

29. $f(x) = \sin \pi x$. Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (R=\infty)$

$$\sin \pi x = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + \cdots = \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1}}$$

$(R=\infty)$

Can do it faster like this: Since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1}}$

31. $f(x) = e^x + e^{2x}$. Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (R=\infty)$,

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \quad (R=\infty)$$

So $e^x + e^{2x} = 2 + 3x + \frac{5x^2}{2!} + \frac{9x^3}{3!} + \frac{17x^4}{4!} + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(2^n+1)x^n}{n!}} \quad (R=\infty)$

33. $f(x) = x\cos(\frac{1}{2}x^2)$. Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (R=\infty)$,

$$\cos(\tfrac{1}{2}x^2) = 1 - \frac{(\frac{1}{2}x^2)^2}{2!} + \frac{(\frac{1}{2}x^2)^4}{4!} - \frac{(\frac{1}{2}x^2)^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2}x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{2^{2n}(2n)!} \quad (R=\infty)$$

So $x\cos(\frac{1}{2}x^2) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2^{2n}(2n)!}} \quad (R=\infty)$

$$= 1 - \frac{x^5}{2^2 \cdot 2!} + \frac{x^9}{2^4 \cdot 4!} - \frac{x^{13}}{2^6 \cdot 6!} + \cdots$$

35. Prepare to use the binomial series formula

$$f(x)=\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4(1+\frac{x^2}{4})}} = \frac{x}{2\sqrt{1+\frac{x^2}{4}}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}}$$

The binomial series is: $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = \binom{k}{0}+\binom{k}{1}x+\binom{k}{2}x^2+\binom{k}{3}x^3+\binom{k}{4}x^4+\binom{k}{5}x^5+\cdots$

$$= 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\frac{k(k-1)(k-2)(k-3)}{4!}x^4+\frac{k(k-1)(k-2)(k-3)(k-4)}{5!}x^5+\cdots \quad (|x|<1)$$

To expand $\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}}$, use the formula above with "x" $=\frac{x^2}{4}$ and $k=-\frac{1}{2}$:

$$\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}} = 1-\frac{1}{2}\cdot\frac{x^2}{4}+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\cdot\left(\frac{x^2}{4}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\cdot\left(\frac{x^2}{4}\right)^3+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}\cdot\left(\frac{x^2}{4}\right)^4+\cdots$$

$$= \underbrace{1}_{n=0}-\underbrace{\frac{1}{2\cdot 4}}_{n=1}x^2+\underbrace{\frac{1\cdot 3}{2^2\cdot 4^2\cdot 2!}}_{n=2}x^4-\underbrace{\frac{1\cdot 3\cdot 5}{2^3\cdot 4^3\cdot 3!}}_{n=3}x^6+\underbrace{\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4^4\cdot 4!}}_{n=4}x^8-\underbrace{\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^5\cdot 4^5\cdot 5!}}_{n=5}x^{10}+\cdots$$

$$= 1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}{\underbrace{2^n\cdot 4^n}_{=8^n=(2^3)^n=2^{3n}}\cdot n!}x^{2n} = 1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}{2^{3n}\cdot n!}x^{2n}$$

Now:

$$\frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}} = \frac{x}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}{2^{3n}\cdot n!}x^{2n}\right] = \boxed{\frac{x}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}{2^{3n+1}\cdot n!}x^{2n+1}}$$

The series converges for $\underbrace{\left|\frac{x^2}{4}\right|<1}$ $\Rightarrow |x^2|<4 \Rightarrow |x|<2$, so $R=2$.

(The reason for this is that the standard binomial series $(1+x)^k$ converges for $|x|<1$, and we subsituted $\frac{x^2}{4}$ in place of x.)

Note: We can't just put $\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}{2^n\cdot 4^n\cdot n!}x^{2n}$ because when $n=0$ using this formula, the $(2n-1)$ factor is $2\cdot 0-1=-1$, which does not give us the correct first term.

37. $f(x) = \sin^2 x = \frac{1}{2}(1-\cos 2x)$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (R=\infty)$$

So, $$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \frac{2^8}{8!}x^8 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n}$$

and, $$1-\cos 2x = 1 - \left(1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots\right) = \frac{2^2}{2!}x^2 - \frac{2^4}{4!}x^4 + \frac{2^6}{6!}x^6 - \frac{2^8}{8!}x^8 + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}}{(2n)!} x^{2n}$$

Finally, $$\frac{1}{2}(1-\cos 2x) = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}}{(2n)!} x^{2n} = \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} x^{2n}} \quad (R=\infty)$$

47. Evaluate the definite integral as an infinite series. $\int x\cos(x^3)\,dx$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (R=\infty)$$

$$\cos(x^3) = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \frac{(x^3)^8}{8!} - \cdots \quad (R=\infty)$$

$$= 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \frac{x^{24}}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

$$x\cos(x^3) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \frac{x^{25}}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \quad (R=\infty)$$

$$\int x\cos(x^3)\,dx = C + \frac{x^2}{2} - \frac{x^8}{2!\cdot 8} + \frac{x^{14}}{4!\cdot 14} - \frac{x^{20}}{6!\cdot 20} + \frac{x^{26}}{8!\cdot 26} - \cdots = \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(2n)!\,(6n+2)}}$$

$(R=\infty)$

51. Use series to approximate $\int_0^1 x\cos(x^3)\,dx$ to within 3 decimal places accuracy.

$$\int_0^1 x\cos(x^3)\,dx = \frac{x^2}{2} - \frac{x^8}{2!\cdot 8} + \frac{x^{14}}{4!\cdot 14} - \frac{x^{20}}{6!\cdot 20} + \frac{x^{26}}{8!\cdot 26} - \cdots \Big]_0^1$$

$$= \frac{1}{2} - \frac{1}{2\cdot 8} + \frac{1}{24\cdot 14} - \underbrace{\frac{1}{720\cdot 20}}_{\approx 7\times 10^{-5} < .001} + \cdots \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{24\cdot 14} \approx \boxed{.440}$$

11.10 homework (more practice)

49. Evaluate the indefinite integral as an infinite series. $\int \frac{\cos x - 1}{x}\,dx$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \quad (R=\infty)$$

$$\cos x - 1 = \frac{-x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \quad (R=\infty)$$

$$\frac{\cos x - 1}{x} = -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \cdots \quad (R=\infty)$$

$$\int \frac{\cos x - 1}{x}\,dx = C - \frac{x^2}{2!\cdot 2} + \frac{x^4}{4!\cdot 4} - \frac{x^6}{6!\cdot 6} + \frac{x^8}{8!\cdot 8} - \cdots = \boxed{C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!\cdot(2n)}} \quad (R=\infty)$$

43. Use the Maclaurin series for e^x to calculate $e^{-0.2}$ correct to five decimal places.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \text{ so } e^{-0.2} = \underbrace{1 - .2 + \frac{.2^2}{2!} - \frac{.2^3}{3!} + \frac{.2^4}{4!}}_{\text{add these terms}} - \underset{\substack{\uparrow \\ 3\times10^{-6}}}{\frac{.2^5}{5!}} + \underset{\substack{\uparrow \\ 9\times10^{-8}}}{\frac{.2^6}{6!}}$$

$$\approx \boxed{.81873}$$

41. Find the Maclaurin series of $f(x) = xe^{-x}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (R=\infty)$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$xe^{-x} = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \cdots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n!}}$$

55. Use series to evaluate the limit. $\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3}$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$x - \tan^{-1}x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \cdots$$

$$\frac{x - \tan^{-1}x}{x^3} = \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \frac{x^6}{9} + \cdots$$

$$\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3} = \boxed{\frac{1}{3}}$$

25. Use the binomial series to expand the function as a power series, $\sqrt{1+x}$

$\sqrt{1+x} = (1+x)^{\frac{1}{2}}$. Since $(1+x)^k = \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \cdots$ Converges for $|x|<1$ $(R=1)$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots$$

$\boxed{k=\frac{1}{2}}$

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{5!}x^5 + \cdots$$

$$= 1 + \frac{1}{2^1}x - \frac{1}{2^2\cdot 2!}x^2 + \frac{1\cdot 3}{2^3\cdot 3!}x^3 - \frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}x^5 + \cdots$$

$$= \boxed{1 + \tfrac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdot 7\cdot \cdots \cdot (2n-3)}{2^n \cdot n!} x^n} \quad (R=1)$$

27. Use the binomial series to expand the function as a power series. $\frac{1}{(2+x)^3}$

$\frac{1}{(2+x)^3} = \frac{1}{\left(2\left(1+\frac{x}{2}\right)\right)^3} = \frac{1}{8\left(1+\frac{x}{2}\right)^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$. Use the binomial series formula for $(1+x)^k$, with "x" $= \frac{x}{2}$ and $k=-3$:

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots$$

$$\left(1+\frac{x}{2}\right)^{-3} = 1 - 3\cdot\frac{x}{2} + \frac{(-3)(-4)}{2!}\cdot\left(\frac{x}{2}\right)^2 + \frac{(-3)(-4)(-5)}{3!}\cdot\left(\frac{x}{2}\right)^3 + \frac{(-3)(-4)(-5)(-6)}{4!}\cdot\left(\frac{x}{2}\right)^4 + \cdots$$

$$= 1 - \frac{3}{2^1}x + \frac{3\cdot 4}{2^2\cdot 2!}x^2 - \frac{3\cdot 4\cdot 5}{2^3\cdot 3!}x^3 + \frac{3\cdot 4\cdot 5\cdot 6}{2^4\cdot 4!}x^4 - \frac{3\cdot 4\cdot 5\cdot 6\cdot 7}{2^5\cdot 5!}x^5 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{3\cdot 4\cdot 5\cdot \cdots \cdot (n+2)}{2^n \cdot n!} x^n$$

So $\frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8} + \sum_{n=1}^{\infty} (-1)^n \frac{3\cdot 4\cdot 5\cdot \cdots \cdot (n+2)}{2^{n+3}\cdot n!} x^n = \frac{1}{8} + \sum_{n=1}^{\infty} (-1)^n \frac{\cancel{2\cdot 3\cdot 4\cdot 5\cdots n}\,(n+1)(n+2)}{2^{n+4}\,\cancel{n!}} x^n$

in an effort to simplify, • the numerator and denominator by 2.

$$= \boxed{\frac{1}{8} + \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2^{n+4}} x^n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2^{n+4}} x^n}$$