1. a. A <u>sequence</u> is an ordered list of numbers.

b. $\lim\limits_{n\to\infty} a_n = 8$ means that the terms of the sequence approach 8 as $n \to \infty$.

This means that the sequence $\{a_n\}$ converges.

c. $\lim\limits_{n\to\infty} a_n = \infty$ means that the terms of the sequence approach ∞ (increase without bound) as $n \to \infty$. This means that the sequence diverges.

2. a. If $\lim\limits_{n\to\infty} a_n$ exists, then the sequence $\{a_n\}$ converges. Exs: $a_n = \frac{1}{n}$, $b_n = 1 - \frac{1}{4^n}$.

This means $\lim\limits_{n\to\infty} a_n = L$ (where L is a single, finite number).

b. If $\lim\limits_{n\to\infty} a_n$ does not exist or is ∞, then the sequence $\{a_n\}$ diverges.

Exs: $a_n = \sin n$, $b_n = 3^n$.

3. $a_n = 1 - (0.2)^n$ $\quad a_1 = 1 - 0.2^1 = .8 \quad a_2 = 1 - 0.2^2 = .96 \quad a_3 = 1 - .2^3 = .992$

$a_4 = 1 - .2^4 = .9984 \quad a_5 = .99968$

5. $a_n = \frac{3(-1)^n}{n!}$ $\quad a_1 = \frac{3(-1)^1}{1!} = -3 \quad a_2 = \frac{3(-1)^2}{2\cdot 1} = \frac{3}{2} \quad a_3 = \frac{\cancel{3}(-1)^3}{\cancel{3}\cdot 2\cdot 1} = -\frac{1}{2}$

$a_4 = \frac{\cancel{3}(-1)^4}{4\cdot\cancel{3}\cdot 2\cdot 1} = \frac{1}{8} \quad a_5 = \frac{\cancel{3}(-1)^5}{5\cdot 4\cdot\cancel{3}\cdot 2\cdot 1} = -\frac{1}{40}$

7. $a_1 = 3,\ a_{n+1} = 2a_n - 1 \quad a_2 = 2a_1 - 1 = 2\cdot 3 - 1 = 5 \quad a_3 = 2a_2 - 1 = 2\cdot 5 - 1 = 9$

$a_4 = 2a_3 - 1 = 2\cdot 9 - 1 = 17 \quad a_5 = 2a_4 - 1 = 2\cdot 17 - 1 = 33$

9. $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\}$ $\quad a_n = \frac{1}{2n-1}$

$a_1 \; a_2 \; a_3 \; a_4 \; a_5$

11. $\{2, 7, 12, 17, 22, 27, \ldots\}$ $\quad a_n = 5n - 3$

$a_1 \; a_2 \; a_3 \; a_4 \; a_5 \; a_6$

13. $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, \ldots\}$ $\quad a_n = \dfrac{(-1)^{n+1}\,2^{n-1}}{3^{n-1}}$ OR $\dfrac{(-1)^{n-1}\,2^{n-1}}{3^{n-1}}$ OR $\left(-\frac{2}{3}\right)^{n-1}$.

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

15. $a_n = \dfrac{n}{2n+1}$ $\quad a_1 = \dfrac{1}{2\cdot1+1} = \dfrac{1}{3}$ $\quad a_2 = \dfrac{2}{2\cdot2+1} = \dfrac{2}{5}$ $\quad a_3 = \dfrac{3}{2\cdot3+1} = \dfrac{3}{7}$ $\quad a_4 = \dfrac{4}{2\cdot4+1} = \dfrac{4}{9}$

$a_5 = \dfrac{5}{11}$ $\quad a_6 = \dfrac{6}{13}$ Appears that the terms are approaching $\frac{1}{2}$.

Proof: $\displaystyle\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{\left(\frac{n}{n}\right)}{\left(\frac{2n+1}{n}\right)} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2}.$

* Shortcut: When the numerator and denominator have the same degree, the limit as $n\to\infty$ will equal the ratio of the leading coefficients. i.e. $\displaystyle\lim_{n\to\infty} \frac{3n^4+5n+4}{7n^4-2n^3+1} = \frac{3}{7}$.

17. $a_n = 1-(0.2)^n$ $\quad \displaystyle\lim_{n\to\infty} 1-(0.2)^n = 1-0 = \boxed{1.}$ $\boxed{\text{Converges}}$ $\left[0.2<1, \text{ and } \displaystyle\lim_{n\to\infty} r^n = 0 \text{ for } |r|<1\right]$ (This is Theorem 9 on page 681)

Ratio of leading coefficients: $\frac{5}{1} = 5$.

19. $a_n = \dfrac{3+5n^2}{n+n^2}$. $\quad \displaystyle\lim_{n\to\infty} \frac{5n^2+3}{n^2+n} = \boxed{5}$ $\boxed{\text{Converges}}$

[Long way: divide the num. and den. by the highest power of n that occurs in the den. (n^2) $\displaystyle\lim_{n\to\infty} \frac{5n^2+3}{n^2+n} = \lim_{n\to\infty} \frac{\left(\frac{5n^2+3}{n^2}\right)}{\left(\frac{n^2+n}{n^2}\right)} = \lim_{n\to\infty} \frac{5+\frac{3}{n^2}}{1+\frac{1}{n}} = \frac{5+0}{1+0} = \frac{5}{1} = 5.$ Converges.]

21. $a_n = e^{\frac{1}{n}}$ $\quad \displaystyle\lim_{n\to\infty} e^{\frac{1}{n}} = e^0 = \boxed{1 \text{ Converges}}$

23. $a_n = \tan\left(\dfrac{2n\pi}{1+8n}\right)$. $\quad \displaystyle\lim_{n\to\infty} \tan\left(\frac{2\pi n}{8n+1}\right) = \tan\frac{2\pi}{8} = \tan\frac{\pi}{4} = \boxed{1 \text{ Converges}}$

see below for long way

25. $a_n = \dfrac{(-1)^{n-1}\,n}{n^2+1}$ $\quad \displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n^2+1} = 0.$ Therefore,

by Theorem 6 page 679, since we know $\displaystyle\lim_{n\to\infty} |a_n| = 0$, we must have $\displaystyle\lim_{n\to\infty} a_n = \boxed{0.}$

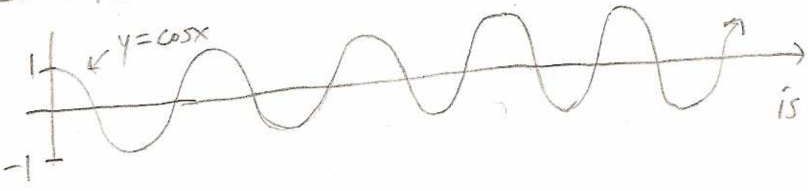
Thus, $\{a_n\}$ $\boxed{\text{converges.}}$

$\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\left(\frac{n}{n^2}\right)}{\left(\frac{n^2+1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n}} = \frac{0}{1+0} = 0.$

11.1 homework

27. $a_n = \cos\left(\frac{n}{2}\right)$. $\lim_{n\to\infty} \cos\left(\frac{n}{2}\right) = \cos\infty$, which does not exist $\Rightarrow$ $\boxed{\text{Diverges}}$

since the cosine oscillates infinitely between the values -1 and 1 as the argument increases.



as $x\to\infty$, the cosine graph is not approaching a fixed #.

29. $\left\{\frac{(2n-1)!}{(2n+1)!}\right\}$ $a_n = \frac{\cancel{(2n-1)(2n-2)(2n-3)\cdots3\cdot2\cdot1}}{(2n+1)(2n)\cancel{(2n-1)(2n-2)(2n-3)\cdots3\cdot2\cdot1}} = \frac{1}{(2n+1)(2n)}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(2n+1)(2n)} = \boxed{0.\ \text{Converges}}$

31. $\left\{\frac{e^n+e^{-n}}{e^{2n}-1}\right\}$ $a_n = \frac{e^n+e^{-n}}{e^{2n}-1} = \frac{\left(\frac{e^n+e^{-n}}{e^{2n}}\right)}{\left(\frac{e^{2n}-1}{e^{2n}}\right)} = \frac{\frac{1}{e^n}+\frac{1}{e^{3n}}}{1-\frac{1}{e^{2n}}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{e^n}+\frac{1}{e^{3n}}}{1-\frac{1}{e^{2n}}} = \frac{0+0}{1-0} = \frac{0}{1} = \boxed{0.\ \text{Converges}}$

OR:

$a_n = \frac{\left(\frac{e^n+e^{-n}}{e^n}\right)}{\left(\frac{e^{2n}-1}{e^n}\right)} = \frac{1+\frac{1}{e^{2n}}}{e^n-\frac{1}{e^n}}$, so $\lim_{n\to\infty} \frac{1+\frac{1}{e^{2n}}}{e^n-\frac{1}{e^n}} = \frac{1+0}{\infty-0} = \frac{1}{\infty} = \boxed{0.\ \text{Converges.}}$

33. $\{n^2e^{-n}\}$. $\lim_{n\to\infty} n^2e^{-n} = \lim_{n\to\infty} \frac{n^2}{e^n} = \frac{\infty}{\infty}$ Indeterminate $\Rightarrow$ need L'Hospital's rule.

$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$ still Indeterminate $\Rightarrow$ need to use L'Hospital again

$\overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0.$

Therefore (by theorem 3 page 678) $\lim_{n\to\infty} \frac{n^2}{e^n} = \boxed{0.\ \text{Converges}}$

35. $a_n = \frac{\cos^2 n}{2^n}$. Note that $0 \le \cos^2 n \le 1$, so $0 \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}$.

$\lim_{n\to\infty} \frac{1}{2^n} = 0$, so by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = \boxed{0.\ \text{Converges}}$

37. $a_n = n\sin\left(\frac{1}{n}\right)$. $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \infty\cdot 0$ Indeterminate $\Rightarrow$ L'Hospital needed

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \overset{(H)}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\left(\cancel{-\frac{1}{x^2}}\right)}{\left(\cancel{-\frac{1}{x^2}}\right)} = \lim_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1.$$

$\frac{0}{0}$ ready for L'Hospital

Therefore (by Thrm. 3), $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \boxed{1.\ \text{Converges}}$

39. $a_n = \left(1+\frac{2}{n}\right)^n$. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = 1^\infty$ Indeterminate $\Rightarrow$ need L'Hospital

$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = ?$ Let $y = \left(1+\frac{2}{x}\right)^x$. Then $\ln y = x\ln\left(1+\frac{2}{x}\right) = \frac{\ln\left(1+\frac{2}{x}\right)}{\left(\frac{1}{x}\right)}$.

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{\left[\frac{1}{1+\frac{2}{x}}\right]\cdot\left(-\frac{2}{\cancel{x^2}}\right)}{\left(-\frac{1}{\cancel{x^2}}\right)}$$

$\frac{0}{0}$ ready for L'Hospital

$\frac{d}{dx}\left(\frac{2}{x}\right) = \frac{d}{dx}\left(2x^{-1}\right) = -2x^{-2} = \frac{-2}{x^2}$

$$= 2\cdot\lim_{x\to\infty}\left[\frac{1}{1+\frac{2}{x}}\right] = 2\cdot\frac{1}{1+0} = 2\cdot 1 = 2. \quad \therefore \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^2.$$

Therefore (by Thrm. 3), $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \boxed{e^2.\ \text{Converges}}$

41. $a_n = \ln(2n^2+1) - \ln(n^2+1) = \ln\left(\frac{2n^2+1}{n^2+1}\right)$. $\lim_{n\to\infty}\ln\left(\frac{2n^2+1}{n^2+1}\right) = \boxed{\ln 2.\ \text{Convergent}}$

43. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$ $\boxed{\text{diverges}}$ since the sequence will continue to alternate between 0 and 1 as $n\to\infty$.

45. $a_n = \frac{n!}{2^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n-2)(n-1)n}{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2\cdot 2} \underbrace{\ge \frac{1}{2}\cdot\frac{n}{2}}_{\text{for } n>1} = \frac{n}{4} \to \infty$ as $n\to\infty$.

So, since $\lim_{n\to\infty}\frac{n!}{2^n} \ge \lim_{n\to\infty}\frac{n}{4} = \infty$, we have $\lim_{n\to\infty}\frac{n!}{2^n} = \infty$. $\boxed{\text{Diverges}}$

55. $a_n = 1000(1.06)^n$. $\{a_n\} = \{1060, 1123.6, 1191.02, 1262.48, 1338.23, \ldots\}$

The sequence is divergent, since $\lim_{n\to\infty} 1000(1.06)^n = \infty$.

$[1.06 > 1$, and $\lim_{n\to\infty} r^n = \infty$ when $r > 1]$

61. $a_n = \frac{1}{2n+3}$ $a_1 = \frac{1}{5}$, $a_2 = \frac{1}{7}$, $a_3 = \frac{1}{9}$, ... It appears to be decreasing.

Check: $f(x) = \frac{1}{2x+3} = (2x+3)^{-1}$. $f'(x) = -(2x+3)^{-2} \cdot 2 = \frac{-2}{(2x+3)^2} < 0$ for all $x \geq 1$. ✓

Another way to check: $a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5} < \frac{1}{2n+3} = a_n$, so $\{a_n\}$ is decreasing since $a_{n+1} < a_n$ for $n \geq 1$. ✓

$\lim_{n\to\infty} \frac{1}{2n+3} = 0 \Rightarrow \{a_n\}$ is bounded below by 0 (and bounded above by $\frac{1}{5}$).

63. $a_n = n(-1)^n$ $a_1 = (-1)^1 \cdot 1 = -1$, $a_2 = (-1)^2 \cdot 2 = 2$, $a_3 = -3$, $a_4 = 4$, $a_5 = -5$, etc.

This sequence is neither increasing nor decreasing since its terms alternate in sign, so it is not monotonic. Since $\lim_{n\to\infty} n = \infty$, the sequence is not bounded.

65. $a_n = \frac{n}{n^2+1}$ First few terms: $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \ldots$ appears to be decreasing.

Check: $f(x) = \frac{x}{x^2+1}$ $f'(x) = \frac{x^2+1-x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$ if $1-x^2 < 0 \Leftrightarrow 1 < x^2$
$\Leftrightarrow x^2 > 1$
$\Leftrightarrow x > 1$. ✓

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, so the sequence is bounded below by 0 (and above by $\frac{1}{2}$).