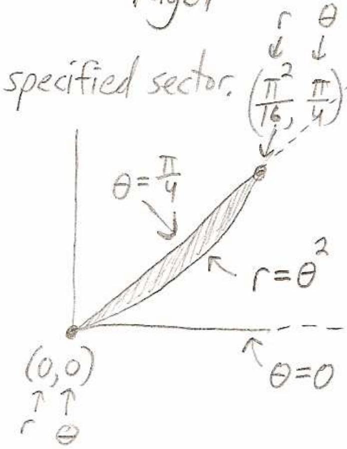
10.4 homework areas in polar coordinates

1. Find the area of the region that is bounded by the curve and lies in the specified sector.

$r=\theta^2,\ 0\le\theta\le\frac{\pi}{4}$

$$A=\int_a^b \frac{1}{2}r^2\,d\theta=\int_0^{\frac{\pi}{4}}\frac{1}{2}(\theta^2)^2\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\theta^4\,d\theta=\frac{1}{2}\cdot\frac{\theta^5}{5}\Big]_0^{\frac{\pi}{4}}=\frac{1}{10}\cdot\frac{\pi^5}{4^5}$$

$$=\boxed{\frac{\pi^5}{10{,}240}}$$



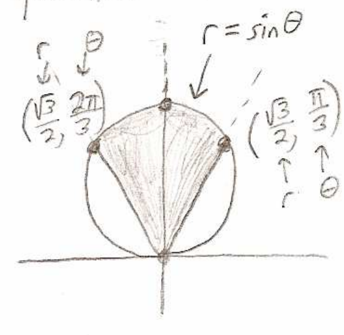
3. Find the area of the region that is bounded by the curve and lies in the specified sector.

$r=\sin\theta,\ \frac{\pi}{3}\le\theta\le\frac{2\pi}{3}$

(Can just do twice the area in the 1st quadrant)

$$A=\int_a^b\frac{1}{2}r^2\,d\theta=\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{1}{2}\sin^2\theta\,d\theta=\cancel{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cancel{\tfrac{1}{2}}\sin^2\theta\,d\theta$$

θ	r
0	0
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}=.87$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}=.87$
π	0



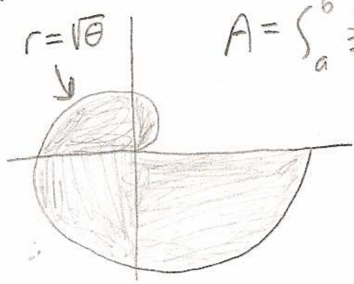
$$=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{2}(1-\cos2\theta)\,d\theta=\frac{1}{2}\left[\theta-\frac{1}{2}\sin2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}-\frac{1}{2}\cdot0\right)-\left(\frac{\pi}{3}-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]=\frac{1}{2}\left[\frac{\pi}{2}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]=\frac{1}{2}\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right]=\boxed{\frac{\pi}{12}+\frac{\sqrt{3}}{8}}$$

5. Find the area of the shaded region. $(r=\sqrt{\theta},\ 0\le\theta\le2\pi)$

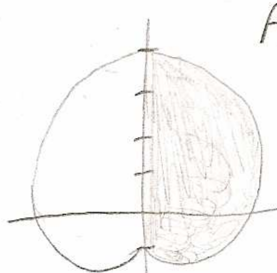
$$A=\int_a^b\frac{1}{2}r^2\,d\theta=\int_0^{2\pi}\frac{1}{2}(\sqrt{\theta})^2\,d\theta=\frac{1}{2}\int_0^{2\pi}\theta\,d\theta=\frac{1}{2}\cdot\frac{\theta^2}{2}\Big]_0^{2\pi}=\frac{1}{4}\cdot4\pi^2$$

$$=\boxed{\pi^2}$$



7. Find the area of the shaded region. $(r=4+3\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2})$

$$A=\int_a^b\frac{1}{2}r^2\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}(4+3\sin\theta)^2\,d\theta=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(16+24\sin\theta+9\sin^2\theta)\,d\theta$$



$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(16+24\sin\theta+9\cdot\frac{1}{2}[1-\cos2\theta]\right)d\theta$$

$16+\frac{9}{2}=\frac{32}{2}+\frac{9}{2}=\frac{41}{2}$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(\frac{41}{2}+24\sin\theta-\frac{9}{2}\cos2\theta\right)d\theta=\frac{1}{2}\left[\frac{41}{2}\theta-24\cos\theta-\frac{9}{2}\frac{\sin2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[\left(\frac{41\pi}{4}-0-0\right)-\left(-\frac{41\pi}{4}-0-0\right)\right]=\frac{1}{2}\left[\frac{41\pi}{4}+\frac{41\pi}{4}\right]=\frac{1}{\cancel{2}}\cdot\frac{\overset{41}{\cancel{82}}\pi}{4}=\boxed{\frac{41\pi}{4}}$$

Just some thoughts on how #7 can be done a little faster.

Note: $\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (16+24\sin\theta+9\sin^2\theta)\,d\theta$ can simplify to $\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (16+9\sin^2\theta)\,d\theta$

because $24\sin\theta$ is an odd function and $\int_{-a}^{a}$ of an odd function is 0. Also,

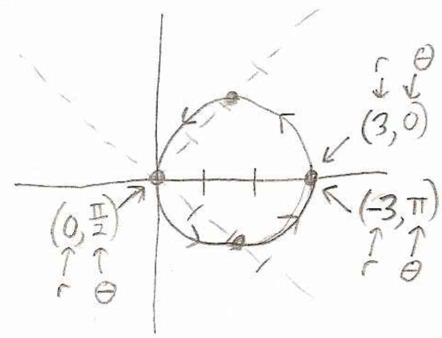
$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (16+9\sin^2\theta)\,d\theta$ is the same as $\frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}}(16+9\sin^2\theta)\,d\theta$ because $16+9\sin^2\theta$

is an even function and $\int_{-a}^{a}$ of an even function is $2\cdot\int_0^a$ of the even function.

9. Sketch the curve and find the area that it encloses. $r=3\cos\theta$ (circle, radius $\frac{3}{2}$)

simplest way $\rightarrow A=\pi r^2=\pi\cdot\left(\frac{3}{2}\right)^2=\boxed{\frac{9\pi}{4}}$

θ	r
0	3
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}=2.1$
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	$-\frac{3\sqrt{2}}{2}=-2.1$
π	-3



OR: $A=\int_a^b \frac{1}{2}r^2\,d\theta=\int_0^{\pi}\frac{1}{2}(3\cos\theta)^2\,d\theta$

$=\frac{9}{2}\int_0^{\pi}\cos^2\theta\,d\theta=\frac{9}{2}\cdot\frac{1}{2}\int_0^{\pi}(1+\cos 2\theta)\,d\theta$

$=\frac{9}{4}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi}=\frac{9}{4}\left[(\pi+0)-(0+0)\right]=\boxed{\frac{9\pi}{4}}$

[Could do $A=2\cdot\int_0^{\frac{\pi}{2}}\frac{1}{2}(3\cos\theta)^2\,d\theta=9\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\frac{9}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta=\frac{9}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\boxed{\frac{9\pi}{4}}$]

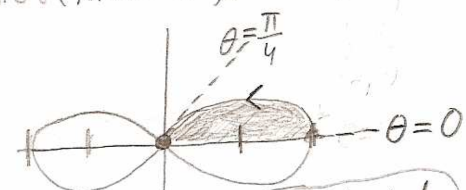
just multiply the area of the top half of the circle by 2

11. Sketch the curve and find the area that it encloses. $r^2=4\cos 2\theta \rightarrow r=\pm 2\sqrt{\cos 2\theta}$

don't need ± to get entire graph

Using the calculator with $r_1=2\sqrt{\cos 2\theta}$, $\theta_{min}=0$, $\theta_{max}=2\pi$, θ step $=.005$, $x_{min}=-2$, $x_{max}=2$, $y_{min}=-2$, $y_{max}=2$, get the graph of a figure 8 ("lemniscate"):

To simplify matters, find the shaded area and multiply by 4.



$A=4\cdot\int_0^{\frac{\pi}{4}}\frac{1}{2}r^2\,d\theta=4\cdot\int_0^{\frac{\pi}{4}}\frac{1}{2}\cdot 4\cos 2\theta\,d\theta$

$=8\int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta=8\cdot\frac{\sin 2\theta}{2}\Big]_0^{\frac{\pi}{4}}=4\cdot(1-0)=\boxed{4}$

θ	r^2	r
0	4	2
$\frac{\pi}{4}$	0	0

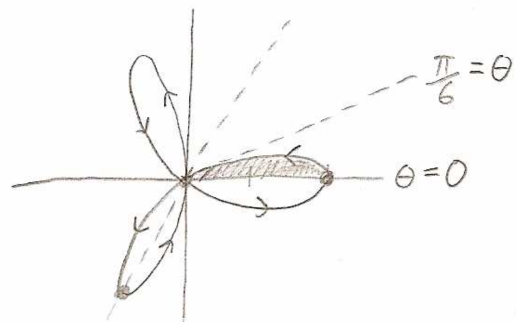
The shaded area is the area bounded by $r^2=4\cos 2\theta$, $0\le\theta\le\frac{\pi}{4}$

13. Sketch the curve and find the area that it encloses. $r = 2\cos 3\theta$ (3-leaved rose)

Find the shaded area and multiply by 6:

θ	r
0	2
$\frac{\pi}{6}$	0
$\frac{\pi}{3}$	-2

etc.



$$A = 6 \cdot \int_0^{\frac{\pi}{6}} \frac{1}{2} r^2 \, d\theta$$

$$= 3 \cdot \int_0^{\frac{\pi}{6}} 4\cos^2 3\theta \, d\theta$$

$$= 12 \cdot \frac{1}{2} \int_0^{\frac{\pi}{6}} (1 + \cos 6\theta) \, d\theta = 6\left[\theta + \frac{\sin 6\theta}{2}\right]_0^{\frac{\pi}{6}}$$

$$= 6\left[\left(\frac{\pi}{6} + 0\right) - (0 + 0)\right] = \boxed{\pi}$$

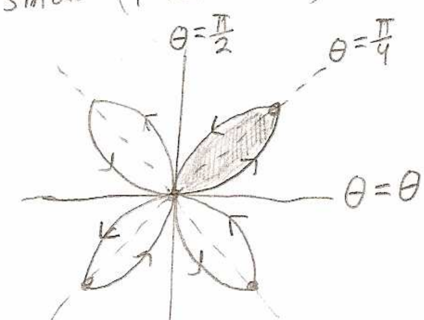
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

so $\cos^2(3\theta) = \frac{1}{2}(1 + \cos 2\cdot 3\theta) = \frac{1}{2}(1 + \cos 6\theta)$

17. Find the area of the region enclosed by one loop of the curve. $r = \sin 2\theta$ (4-leaved rose)

θ	r
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	-1
π	0
$\frac{5\pi}{4}$	1

etc.



The area of the 1st loop is:

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 4\theta) \, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right] = \boxed{\frac{\pi}{8}}$$

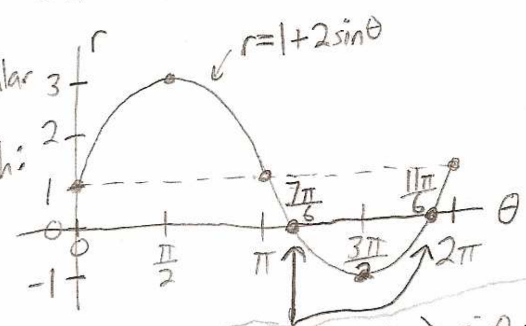
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

so $\sin^2(2\theta) = \frac{1}{2}(1 - \cos 2\cdot 2\theta)$

$= \frac{1}{2}(1 - \cos 4\theta)$

21. Find the area of the region enclosed by one loop of the curve. $r = 1 + 2\sin\theta$ (inner loop)

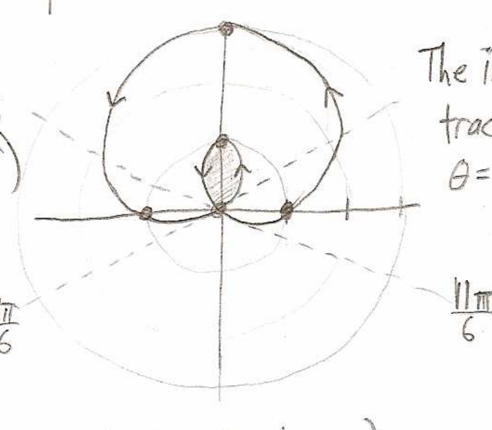
Regular trig graph:



$1 + 2\sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

$\Rightarrow \theta = \boxed{\frac{7\pi}{6}}, \boxed{\frac{11\pi}{6}}$

Polar graph: ("limacon")



The inner loop is traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

θ	r
0	1
$\frac{\pi}{2}$	3
π	1
$\frac{7\pi}{6}$	0
$\frac{3\pi}{2}$	-1
$\frac{11\pi}{6}$	0
2π	1

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21. continued

So, $A=\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}r^2\,d\theta = 2\cdot\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \frac{1}{2}(1+2\sin\theta)^2\,d\theta$ $\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}(1+4\sin\theta+4\sin^2\theta)\,d\theta$

twice the area of the right half of the loop

$$=\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}\left(1+4\sin\theta+4\cdot\frac{1}{2}(1-\cos2\theta)\right)d\theta$$

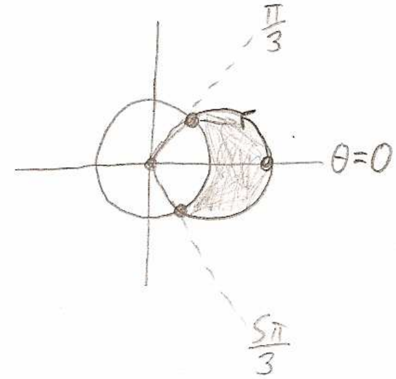
$$=\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}(3+4\sin\theta-2\cos2\theta)\,d\theta = 3\theta-4\cos\theta-\cancel{2}\cdot\frac{\sin2\theta}{\cancel{2}}\Big]_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}$$

$$=\left(3\cdot\frac{3\pi}{2}-4\cdot0-0\right)-\left(3\cdot\frac{7\pi}{6}-4\cdot\frac{-\sqrt{3}}{2}-\sin\frac{7\pi}{3}\right)$$

$$=\frac{9\pi}{2}-\left(\frac{7\pi}{2}+2\sqrt{3}-\frac{\sqrt{3}}{2}\right)=\frac{2\pi}{2}-2\sqrt{3}+\frac{\sqrt{3}}{2}=\pi-\frac{4\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\boxed{\pi-\frac{3\sqrt{3}}{2}}$$

#23-27: Find the area of the region that lies inside the first curve and outside the 2nd curve.

θ	r
0	2
$\frac{\pi}{2}$	0
π	-2
$\frac{\pi}{3}$	1
$\frac{5\pi}{3}$	1



23. $r=2\cos\theta$, $r=1$

curves intersect when $2\cos\theta=1 \Rightarrow \cos\theta=\frac{1}{2}$

$\Rightarrow \theta=\frac{\pi}{3}, \frac{5\pi}{3}$

The shaded area is twice the area between the curves from $\theta=0$ to $\theta=\frac{\pi}{3}$:

$$A=2\cdot\int_0^{\frac{\pi}{3}}\frac{1}{2}(r_{outer}^2-r_{inner}^2)\,d\theta=\int_0^{\frac{\pi}{3}}\left[(2\cos\theta)^2-1^2\right]d\theta=\int_0^{\frac{\pi}{3}}(4\cos^2\theta-1)\,d\theta$$

$$=\int_0^{\frac{\pi}{3}}\left[4\cdot\frac{1}{2}(1+\cos2\theta)-1\right]d\theta=\int_0^{\frac{\pi}{3}}\left[2+2\cos2\theta-1\right]d\theta=\int_0^{\frac{\pi}{3}}\left[1+2\cos2\theta\right]d\theta$$

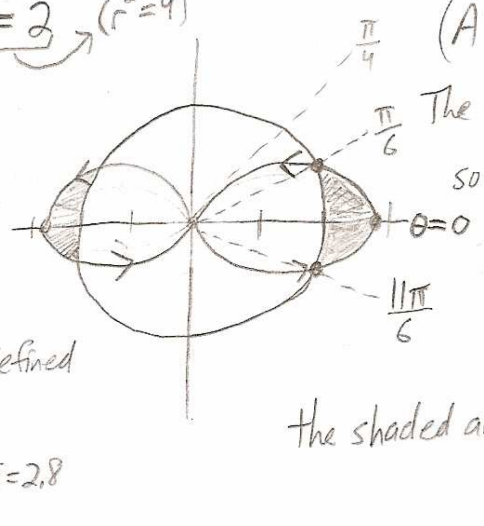
$$=\theta+\cancel{2}\cdot\frac{\sin2\theta}{\cancel{2}}\Big]_0^{\frac{\pi}{3}}=\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)-(0+0)=\boxed{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}$$

25. $r^2=8\cos2\theta$, $r=2$ $(r^2=4)$ (A lemniscate and a circle with radius 2)

$r=2\sqrt{2}\sqrt{\cos2\theta}$

θ	r
0	$2\sqrt{2}=2.8$
$\frac{\pi}{4}$	0
$\frac{\pi}{2}$	undefined
$\frac{3\pi}{4}$	0
π	$2\sqrt{2}=2.8$
$\frac{5\pi}{4}$	0

θ	r
$\frac{3\pi}{2}$	undefined
$\frac{7\pi}{4}$	0
2π	$2\sqrt{2}=2.8$



The curves interesect when $8\cos2\theta=4$

so $\cos2\theta=\frac{1}{2} \Rightarrow 2\theta=\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\Rightarrow \theta=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

The total shaded area is 4 times the shaded area from $\theta=0$ to $\theta=\frac{\pi}{6}$:

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25. continued So, $A = 4\cdot\int_0^{\frac{\pi}{6}} \frac{1}{2}\left[r_{outer}^2 - r_{inner}^2\right] d\theta = 2\int_0^{\frac{\pi}{6}}\left[8\cos 2\theta - 4\right] d\theta$

$$= 2\left[\cancel{8}^{4}\cdot\frac{\sin 2\theta}{\cancel{2}} - 4\theta\right]_0^{\frac{\pi}{6}} = 2\left[\left(4\cdot\sin\frac{\pi}{3} - \frac{4\pi}{6}\right) - (0-0)\right] = 2\left[\cancel{4}^{2}\cdot\frac{\sqrt{3}}{\cancel{2}} - \frac{2\pi}{3}\right] =$$

$$= \boxed{4\sqrt{3} - \frac{4\pi}{3}}$$

27. $r = 3\cos\theta$, $r = 1+\cos\theta$
circle cardioid

circle

θ	r
0	3
$\frac{\pi}{2}$	0
π	-3
etc	

cardioid

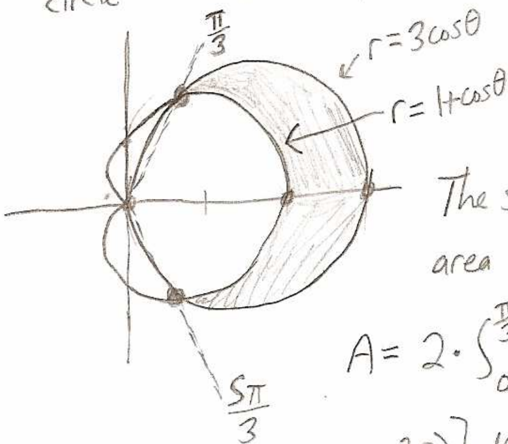
θ	r
0	2
$\frac{\pi}{2}$	1
π	0
etc	

curves intersect when

$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$



The shaded area is twice the shaded area from $\theta = 0$ to $\theta = \frac{\pi}{3}$:

$$A = 2\cdot\int_0^{\frac{\pi}{3}}\frac{1}{2}\left[r_{outer}^2 - r_{inner}^2\right]d\theta = \int_0^{\frac{\pi}{3}}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right]d\theta$$

$$= \int_0^{\frac{\pi}{3}}\left[9\cos^2\theta - (1 + 2\cos\theta + \cos^2\theta)\right]d\theta = \int_0^{\frac{\pi}{3}}\left[8\cos^2\theta - 2\cos\theta - 1\right]d\theta$$

$$= \int_0^{\frac{\pi}{3}}\left[8\cdot\frac{1}{2}(1+\cos 2\theta) - 2\cos\theta - 1\right]d\theta = \int_0^{\frac{\pi}{3}}\left[3 + 4\cos 2\theta - 2\cos\theta\right]d\theta$$

$$= \left[3\theta + \cancel{4}^{2}\,\frac{\sin 2\theta}{\cancel{2}} - 2\sin\theta\right]_0^{\frac{\pi}{3}} = \left(\pi + 2\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2}\right) - (0+0-0) = \boxed{\pi}$$

#29–33: Find the area of the region that lies inside both curves.

29. $r = \sqrt{3}\cos\theta$, $r = \sin\theta$

curves intersect when $\frac{\sqrt{3}\cos\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$

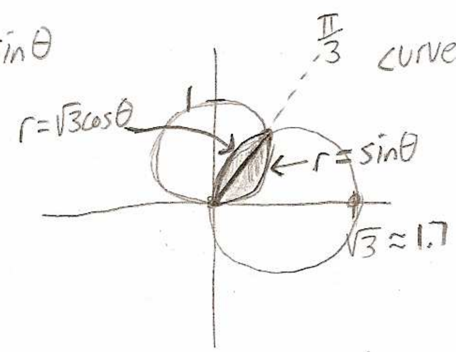
$\Rightarrow \sqrt{3} = \tan\theta \Rightarrow \theta = \frac{\pi}{3}, \frac{4\pi}{3}$

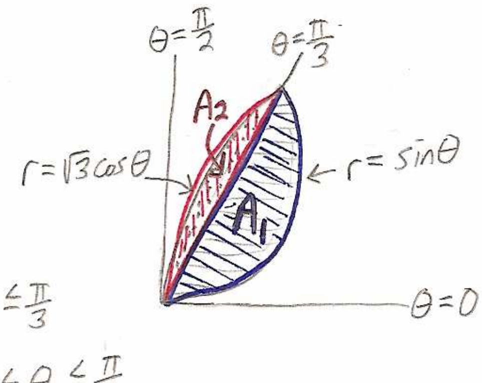
1st circle

θ	r
0	$\sqrt{3}$
$\frac{\pi}{2}$	0
π	$-\sqrt{3}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$

2nd circle

θ	r
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$





From the larger drawing of the shaded region, A_1 = the area bounded by $r = \sin\theta$, $0 \le \theta \le \frac{\pi}{3}$ and A_2 = the area bounded by $r = \sqrt{3}\cos\theta$, $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$.

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29. continued

So, $A_1 = \int_0^{\frac{\pi}{3}} \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\theta\, d\theta = \frac{1}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (1-\cos 2\theta)\, d\theta$

$$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}} = \frac{1}{4}\left[\left(\frac{\pi}{3} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - (0-0)\right] = \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \frac{\pi}{12} - \frac{\sqrt{3}}{16}.$$

And, $A_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 3\cos^2\theta\, d\theta = \frac{3}{2}\cdot\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$

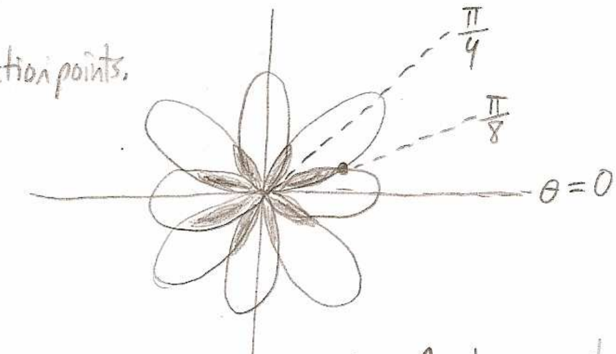
$$= \frac{3}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{3}{4}\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] = \frac{\pi}{8} - \frac{3\sqrt{3}}{16}.$$

Total shaded area $= A_1 + A_2 = \frac{\pi}{12} - \frac{\sqrt{3}}{16} + \frac{\pi}{8} - \frac{3\sqrt{3}}{16} = \frac{2\pi + 3\pi}{24} - \frac{4\sqrt{3}}{16} = \boxed{\frac{5\pi}{24} - \frac{\sqrt{3}}{4}}$

31. $r = \sin 2\theta$, $r = \cos 2\theta$ both are 4-leaved roses. The curves intersect when

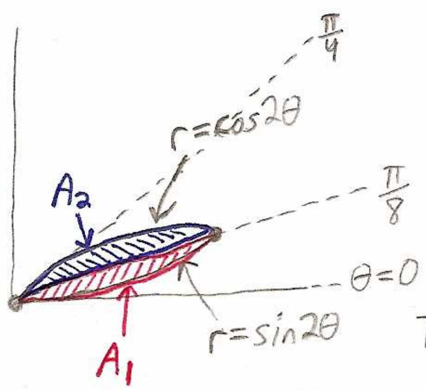
$\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4} \Rightarrow \theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\cancel{\cos 2\theta}}{\cancel{\cos 2\theta}}$



The graph shows that there are several other intersection points.

Below is a larger view of the shaded region in quadrant I centered about $\theta = \frac{\pi}{8}$:



There are 8 such shaded regions. $A_1 = A_2$ by symmetry, so we can find the total shaded area by finding A_1 and multiplying by 16:

Total shaded area $= 16 \cdot A_1 = 16 \cdot \int_0^{\frac{\pi}{8}} \frac{1}{2} r^2\, d\theta$

$$= 8 \cdot \int_0^{\frac{\pi}{8}} \sin^2 2\theta\, d\theta = 8\cdot\frac{1}{2}\int_0^{\frac{\pi}{8}} (1-\cos 4\theta)\, d\theta$$

Note that A_1 is the area of the region bounded by $r = \sin 2\theta$, $0 \le \theta \le \frac{\pi}{8}$.

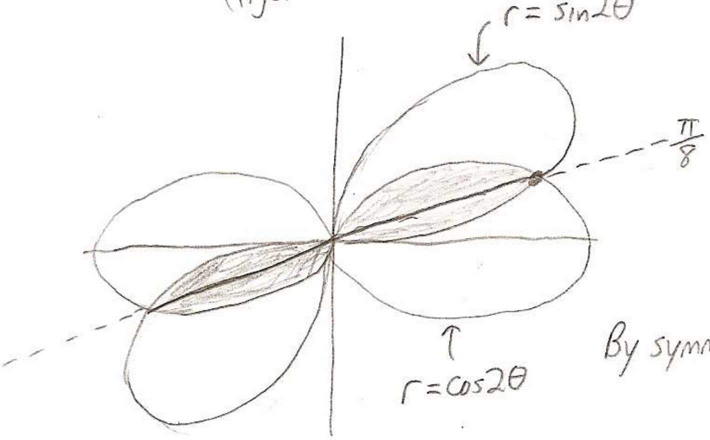
$$= 4\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{8}} = 4\left[\left(\frac{\pi}{8} - \frac{1}{4}\cdot 1\right) - (0-0)\right]$$

$$= 4\left[\frac{\pi}{8} - \frac{1}{4}\right] = \boxed{\frac{\pi}{2} - 1}$$

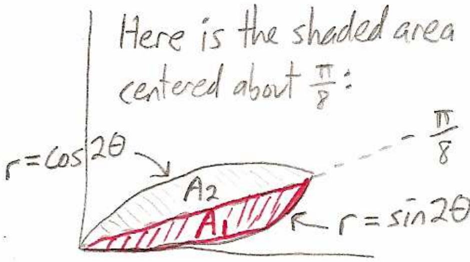
33. $r^2 = \sin 2\theta$, $r^2 = \cos 2\theta$ curves intersect when $\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1$

$\Rightarrow 2\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4} \Rightarrow \theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

two lemniscates (figure 8s)



Here is the shaded area in quadrant I centered about $\frac{\pi}{8}$:



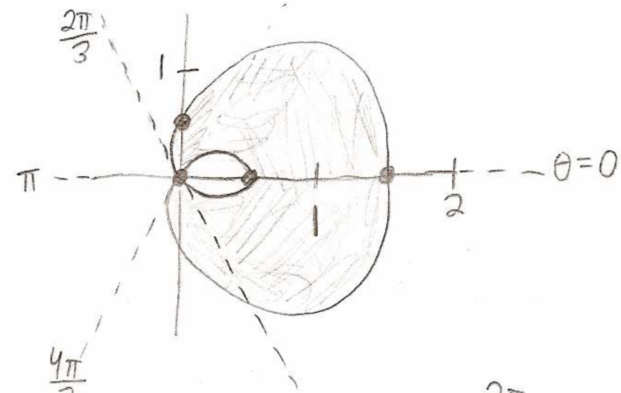
By symmetry, the total shaded area $= 4 \cdot A_1$

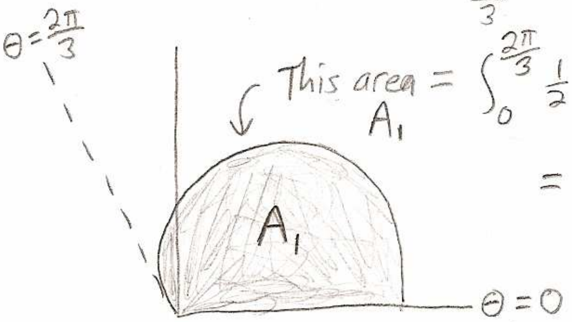
$= 4 \cdot \int_0^{\frac{\pi}{8}} \frac{1}{2} r^2 \, d\theta =$ next line

$$= 2\int_0^{\frac{\pi}{8}} \sin 2\theta \, d\theta = \cancel{2}\left[\frac{-\cos 2\theta}{\cancel{2}}\right]_0^{\frac{\pi}{8}} = -\left[\cos 2\theta\right]_0^{\frac{\pi}{8}} = -\left[\frac{\sqrt{2}}{2} - 1\right] = \boxed{1 - \frac{\sqrt{2}}{2}}$$

35. Find the area inside the larger loop and outside the smaller loop of the limacon $r = \frac{1}{2} + \cos\theta$.

θ	r
0	$\frac{1}{2} + 1 = 1\frac{1}{2}$
$\frac{\pi}{2}$	$\frac{1}{2}$
$\frac{2\pi}{3}$	$\frac{1}{2} - \frac{1}{2} = 0$
π	$\frac{1}{2} - 1 = -\frac{1}{2}$
$\frac{4\pi}{3}$	$\frac{1}{2} - \frac{1}{2} = 0$





This area A_1 $= \int_0^{\frac{2\pi}{3}} \frac{1}{2}\left(\frac{1}{2} + \cos\theta\right)^2 d\theta = \frac{1}{2}\int_0^{\frac{2\pi}{3}} \left(\frac{1}{4} + \cos\theta + \overbrace{\frac{1}{2} + \frac{1}{2}\cos 2\theta}^{\cos^2\theta}\right) d\theta$

$= \frac{1}{2}\int_0^{\frac{2\pi}{3}} \left(\frac{3}{4} + \cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\left[\frac{3}{4}\theta + \sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{2\pi}{3}}$

$$= \frac{1}{2}\left[\left(\frac{\cancel{3}}{4} \cdot \frac{2\pi}{\cancel{3}} + \frac{\sqrt{3}}{2} + \frac{1}{4} \cdot \frac{-\sqrt{3}}{2}\right) - (0+0+0)\right] = \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right] = \frac{1}{2}\left[\frac{\pi}{2} + \frac{3\sqrt{3}}{8}\right] = \overbrace{\frac{\pi}{4} + \frac{3\sqrt{3}}{16}}^{A_1}.$$

The upper half of the inner loop has the same area as the lower half of the inner loop, and the inner loop is traced out as θ sweeps from $\frac{2\pi}{3}$ to π, so the area of the lower half of the inner loop $= A_2 = \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}\left(\frac{1}{2} + \cos\theta\right)^2 d\theta =$ next page...

35. continued

$$A_2 = \frac{1}{2}\left[\frac{3}{4}\theta + \sin\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{2\pi}{3}}^{\pi} = \frac{1}{2}\left[\left(\frac{3\pi}{4}+0+0\right) - \left(\frac{\cancel{3}}{4}\cdot\frac{2\pi}{\cancel{3}} + \frac{\sqrt{3}}{2} + \frac{1}{4}\cdot\frac{-\sqrt{3}}{2}\right)\right]$$

(we just did this integral with different limits of integration)

$$= \frac{1}{2}\left[\frac{3\pi}{4} - \left(\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right)\right] = \frac{1}{2}\left[\frac{3\pi}{4} - \frac{2\pi}{4} - \frac{4\sqrt{3}}{8} + \frac{\sqrt{3}}{8}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] = \frac{\pi}{8} - \frac{3\sqrt{3}}{16} = A_2.$$

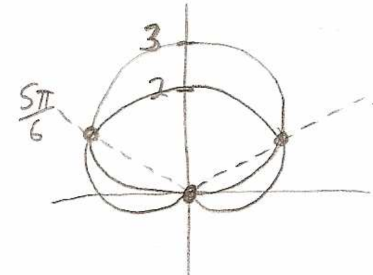
The total shaded area $= 2\cdot A_1 - 2\cdot A_2 = 2\left[\frac{\pi}{4} + \frac{3\sqrt{3}}{16}\right] - 2\left[\frac{\pi}{8} - \frac{3\sqrt{3}}{16}\right] =$

$$= \frac{\pi}{2} + \frac{3\sqrt{3}}{8} - \frac{\pi}{4} + \frac{3\sqrt{3}}{8} = \frac{\pi}{4} + \frac{6\sqrt{3}}{8} = \boxed{\frac{\pi}{4} + \frac{3\sqrt{3}}{4}}$$

#37–41: Find all points of intersection of the given curves.

37. $r = 1+\sin\theta$, $r = 3\sin\theta$

$1+\sin\theta = 3\sin\theta \Rightarrow 1 = 2\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$



θ	r_1	r_2
$\frac{\pi}{6}$	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{5\pi}{6}$	$\frac{3}{2}$	$\frac{3}{2}$

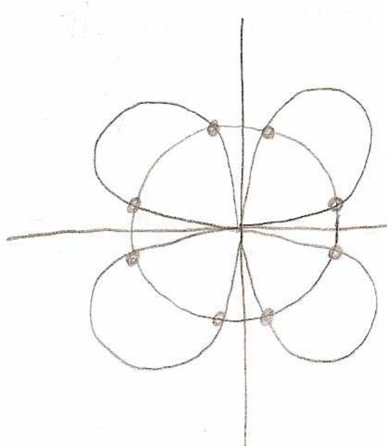
There are 3 intersection points: $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right)$, and the pole (origin).

39. $r = 2\sin 2\theta$, $r = 1$ $\quad 2\sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$. Since $r=1$ is equivalent to $r=-1$, other intersection pts. occur when $2\sin 2\theta = -1 \Rightarrow \sin 2\theta = -\frac{1}{2} \Rightarrow 2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

$\Rightarrow \theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

θ	r
$\frac{\pi}{12}$	$2\sin 2\cdot\frac{\pi}{12} = 2\sin\frac{\pi}{6} = 2\cdot\frac{1}{2} = 1$ ✓ all θs produce $r=1$ or $r=-1$.
$\frac{5\pi}{12}$	$2\sin\frac{5\pi}{6} = 2\cdot\frac{1}{2} = 1$
$\frac{7\pi}{12}$	$2\sin\frac{7\pi}{6} = 2\cdot -\frac{1}{2} = -1$
$\frac{11\pi}{12}$	$2\sin\frac{11\pi}{6} = -1$
$\frac{13\pi}{12}$	$2\sin\frac{13\pi}{6} = 1$
$\frac{17\pi}{12}$	$2\sin\frac{17\pi}{6} = 1$
$\frac{19\pi}{12}$	$2\sin\frac{19\pi}{6} = -1$
$\frac{23\pi}{12}$	$2\sin\frac{23\pi}{6} = -1$



There are 8 intersection points, occurring at the values of θ listed in the table.

10.4 homework

Page 5

41. $r = \sin\theta,\ r = \sin 2\theta$

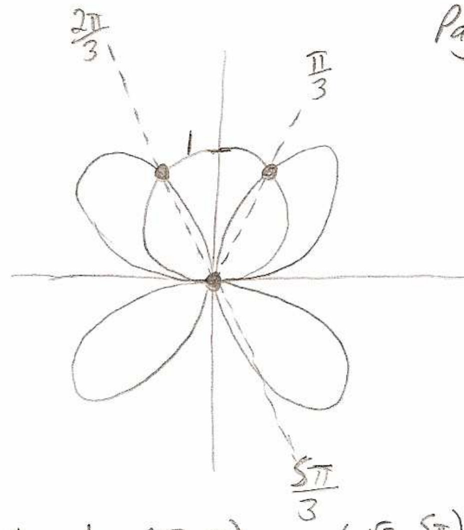
$\sin\theta = \sin 2\theta \Rightarrow \sin\theta = 2\sin\theta\cos\theta$

$\Rightarrow 0 = 2\sin\theta\cos\theta - \sin\theta$

$\Rightarrow 0 = \sin\theta(2\cos\theta - 1)$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$

$\theta = 0, \pi$ $\qquad$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$



θ	r
0	0
π	0
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$

There are 3 intersection points: the pole, $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$, and $\underbrace{\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right)}_{\text{same as } \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)}$.