10.2 homework calculus with parametric curves (★See last page for a list of key formulas) Page 1

1. Find $\frac{dy}{dx}$. $x = t\sin t,\ y = t^2 + t$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{t\cos t + \sin t}$$

3. Find an equation of the tangent to the curve $x = t^4 + 1,\ y = t^3 + t$ at the point corresponding to $t = -1$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2+1}{4t^3}.$$ When $t = -1$, $\frac{dy}{dx} = \frac{3(-1)^2+1}{4(-1)^3} = \frac{4}{-4} = -1.$

t	x	y
-1	2	-2
	x_1	y_1

values for the point-slope formula

$$y - y_1 = m(x - x_1) \Rightarrow y - (-2) = -1(x-2) \Rightarrow y + 2 = -x + 2 \Rightarrow \boxed{y = -x}$$

$y_1 = -2$, $m = \frac{dy}{dx} = -1$, $x_1 = 2$

5. Find an equation of the tangent to the curve $x = e^{\sqrt{t}},\ y = t - \ln t^2$ at the point corresponding to $t = 1$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - \frac{1}{t^2}\cdot 2t}{e^{\sqrt{t}}\cdot \frac{1}{2}t^{-\frac{1}{2}}} = \frac{1 - \frac{2}{t}}{\frac{e^{\sqrt{t}}}{2\sqrt{t}}} = \frac{2\sqrt{t}\left(1 - \frac{2}{t}\right)}{e^{\sqrt{t}}} = \frac{2(-1)}{e} = \frac{-2}{e}.$$

(When $t=1$)

t	x	y
1	e	1
	x_1	y_1

$$y - y_1 = m(x - x_1) \Rightarrow y - 1 = -\frac{2}{e}(x - e)$$
$$\Rightarrow y = -\frac{2}{e}x + 2 + 1 \Rightarrow \boxed{y = -\frac{2}{e}x + 3}$$

7. a. Find an equation of the tangent to the curve $x = 1 + \ln t,\ y = t^2 + 2$ at the point $(1, 3)$ without eliminating the parameter.

When $x = 1$, $1 = 1 + \ln t \Rightarrow t = 1.$
Also when $y = 3$, $3 = t^2 + 2 \Rightarrow t = \pm 1.$
Therefore $t = 1$ corresponds to the point $(1, 3)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{\frac{1}{t}} = 2t^2 = 2.$$ (When $t=1$)

$$y - y_1 = m(x - x_1) \Rightarrow y - 3 = 2(x-1)$$
$$\Rightarrow y = 2x - 2 + 3 \Rightarrow \boxed{y = 2x + 1}$$

b. Find the eq. of the tangent by elim. the parameter.

$x - 1 = \ln t$, so $e^{x-1} = t$. Substituting: $y = \left(e^{x-1}\right)^2 + 2 \Rightarrow y = e^{2x-2} + 2.$ $\frac{dy}{dx} = 2e^{2x-2} = 2\cdot e^0 = 2.$ (When $x=1$)

$$y - y_1 = m(x - x_1) \Rightarrow y - 3 = 2(x-1) \Rightarrow y = 2x - 2 + 3 \Rightarrow \boxed{y = 2x + 1}$$

9. Find an eq. of the tangent(s) to the curve $x = 6\sin t$, $y = t^2 + t$ at the point $(0,0)$. Then graph the curve and the tangent(s).

$x = 0 \Rightarrow 0 = 6\sin t \Rightarrow \sin t = 0 \Rightarrow t = 0, \pi, 2\pi, \ldots = n\pi$

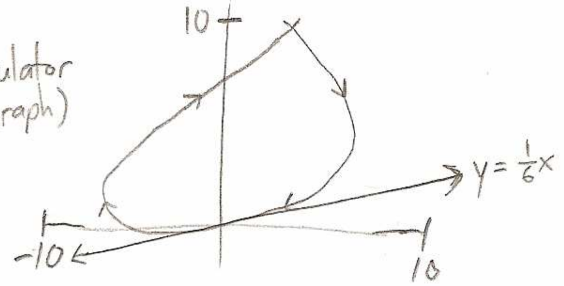
$y = 0 \Rightarrow 0 = t^2 + t \Rightarrow 0 = t(t+1) \Rightarrow t = 0$ or $t = -1$.

Therefore, $t = 0$ corresponds to the point $(0,0)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{6\cos t} = \underset{\text{When } t=0}{\frac{1}{6\cdot 1}} = \frac{1}{6}.$$

$y - y_1 = m(x - x_1) \Rightarrow y - 0 = \frac{1}{6}(x - 0) \Rightarrow \boxed{y = \frac{1}{6}x}$

(Calculator graph)



11. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

$x = 4 + t^2$, $y = t^2 + t^3$ $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$

The formula for $\frac{d^2y}{dx^2}$ is derived from the $\frac{dy}{dx}$ formula by replacing y with $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{so} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

For #11, $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(1 + \frac{3}{2}t\right)}{2t} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t} > 0 \text{ when } t > 0.$$

The curve is concave up for $t > 0$.

13. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

$x = t - e^t$, $y = t + e^{-t}$ $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - e^{-t}}{1 - e^t} = \frac{1 - \frac{1}{e^t}}{1 - e^t} = \frac{e^t - 1}{e^t(1 - e^t)} = \frac{-1}{e^t} = -e^{-t}.$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-e^{-t}\right)}{1 - e^t} = \frac{e^{-t}}{1 - e^t} = \frac{1}{e^t(1 - e^t)}.$$

Since $e^t > 0$ for all t, $\frac{1}{e^t(1-e^t)} > 0$ when $1 - e^t > 0 \Rightarrow 1 > e^t \Rightarrow e^t < 1 \Rightarrow t < 0$.

So, the curve is concave up for $t < 0$.

15. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave up?

$x = 2\sin t,\ y = 3\cos t,\ 0 < t < 2\pi \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\frac{3}{2}\tan t\right)}{2\cos t} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = \frac{-3}{4\cos^3 t} > 0 \text{ when } \cos^3 t < 0$$

$\cos^3 t < 0 \Rightarrow \cos t < 0 \Rightarrow \frac{\pi}{2} < t < \frac{3\pi}{2}$. The curve is concave up for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

17. Find all points on the curve where the tangent is horizontal or vertical.

$x = 10 - t^2,\ y = t^3 - 12t \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 12}{-2t}$. The curve has a horizontal tangent when $\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dt} = 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow 3(t^2 - 4) = 0 \Rightarrow t^2 - 4 = 0 \Rightarrow t = \pm 2$.

There is a vertical tangent when $\frac{dx}{dt} = 0 \Rightarrow -2t = 0 \Rightarrow t = 0$.

t	x	y
-2	6	16
2	6	-16
0	10	0

$\therefore$ At the points $(6,16)$ and $(6,-16)$ there are horizontal tangents, and at the point $(10,0)$ there is a vertical tangent.

19. Find all pts. on the curve $x = 2\cos\theta,\ y = \sin 2\theta$ where the tangent is horiz. or vertical.

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{2}\cos 2\theta}{-\cancel{2}\sin\theta} = \frac{-\cos 2\theta}{\sin\theta}$. There is a horiz. tangent when $\frac{dy}{dx} = 0 \Leftrightarrow$

$$\frac{dy}{d\theta} = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow \left.\begin{cases} 2\theta = \frac{\pi}{2} + 2n\pi \rightarrow \theta = \frac{\pi}{4} + n\pi \\ 2\theta = \frac{3\pi}{2} + 2n\pi \rightarrow \theta = \frac{3\pi}{4} + n\pi \end{cases}\right\} \Rightarrow \theta = \frac{\pi}{4} + \frac{\pi}{2}n$$

θ	x	y
$\frac{\pi}{4}$	$\sqrt{2}$	1
$\frac{3\pi}{4}$	$-\sqrt{2}$	-1
$\frac{5\pi}{4}$	$-\sqrt{2}$	1
$\frac{7\pi}{4}$	$\sqrt{2}$	-1

There are horizontal tangents at the points $(\pm\sqrt{2}, \pm 1)$.

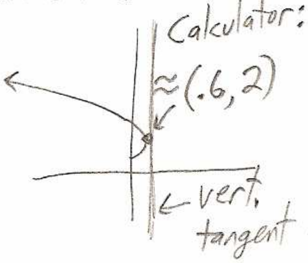
There is a vertical tangent when $\frac{dx}{d\theta} = 0 \Rightarrow \sin\theta = 0$

$\Rightarrow \theta = n\pi$

θ	x	y
0	2	0
π	-2	0

There are vertical tangents at $(\pm 2, 0)$.

21. Use a graph to estimate the coordinates of the rightmost point on the curve $x = t - t^6$, $y = e^t$. Then use calculus to find the exact coordinates.



Since there is a vertical tangent at the rightmost point, we need $\frac{dx}{dt} = 0 \Rightarrow 1 - 6t^5 = 0 \Rightarrow 6t^5 = 1 \Rightarrow t^5 = \frac{1}{6} \Rightarrow t = \sqrt[5]{\frac{1}{6}}$

$t = \frac{1}{\sqrt[5]{6}}$

Exact coordinates: $\boxed{x = \frac{1}{\sqrt[5]{6}} - \frac{1}{6\sqrt[5]{6}},\ y = e^{\frac{1}{\sqrt[5]{6}}}}$

$\approx (.582, 2.011)$

25. Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0,0)$ and find their equations. Sketch the curve. $\to y = \frac{1}{2}\sin 2t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos 2t}{-\sin t}$.

When $x = 0$, $0 = \cos t \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots = \frac{\pi}{2} + n\pi$

When $y = 0$, $0 = \frac{1}{2}\sin 2t \Rightarrow 0 = \sin 2t \Rightarrow 2t = n\pi$

$\Rightarrow t = \frac{\pi}{2} \cdot n$

$= \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots$

The values of t for which both $x = 0$ and $y = 0$ are $t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$t = \frac{\pi}{2} + n\pi$. These t values correspond to the point $(0,0)$ on the curve.

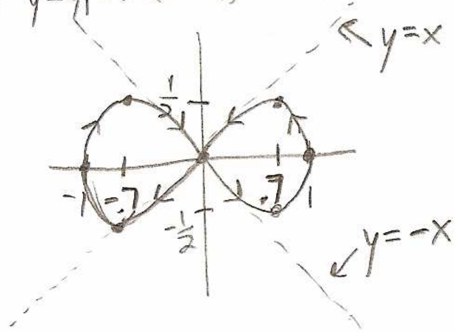
When $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{\cos \cancel{2} \cdot \frac{\pi}{\cancel{2}}}{-\sin \frac{\pi}{2}} = \frac{-1}{-1} = 1$.

When $t = \frac{3\pi}{2}$: $\frac{dy}{dx} = \frac{\cos \cancel{2} \cdot \frac{3\pi}{\cancel{2}}}{-\sin \frac{3\pi}{2}} = \frac{-1}{-(-1)} = -1$.

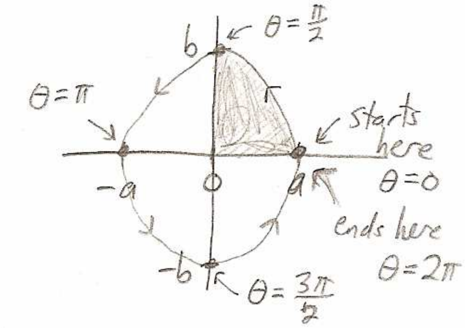
Equation of 1st tangent: $y - y_1 = m(x - x_1) \Rightarrow y - 0 = 1(x - 0) \Rightarrow \boxed{y = x}$

Equation of 2nd tangent: $y - y_1 = m(x - x_1) \Rightarrow y - 0 = -1(x - 0) \Rightarrow \boxed{y = -x}$

t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx .7$	$\frac{1}{2}$
$\frac{\pi}{2}$	0	0
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -.7$	$-\frac{1}{2}$
π	-1	0
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} = -.7$	$\frac{1}{2}$
$\frac{3\pi}{2}$	0	0
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2} = .7$	$-\frac{1}{2}$



31. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$, to find the area that it encloses.



Total area = 4 · shaded area = $4 \cdot \int_0^a y\,dx$

Note: Integrate from left to right, so $\frac{\pi}{2}$ must be the lower limit and 0 the upper limit. $\rightarrow$ $= 4 \cdot \int_{\frac{\pi}{2}}^{0} \underbrace{b\sin\theta}_{y} \cdot \underbrace{(-a\sin\theta\,d\theta)}_{dx}$

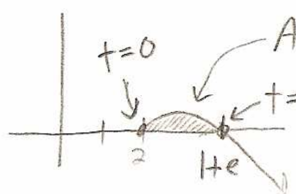
$= -4ab \int_{\frac{\pi}{2}}^{0} \sin^2\theta\,d\theta = 4ab \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)\,d\theta = 2ab \int_0^{\frac{\pi}{2}} (1-\cos 2\theta)\,d\theta$

$= 2ab \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = 2ab\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right] = \pi ab.$

33. Find the area enclosed by the x-axis and the curve $x = 1 + e^t$, $y = t - t^2$.

The curve intersects the x-axis when $y = 0 \Rightarrow t - t^2 = 0 \Rightarrow t(1-t) = 0 \Rightarrow t = 0$ and $t = 1$.

t	x	y
0	2	0
1	1+e	0



Area = $\int_2^{1+e} y\,dx = \int_0^1 \underbrace{(t-t^2)}_{y}\underbrace{(e^t\,dt)}_{dx}$

$= \int_0^1 te^t\,dt - \int_0^1 t^2e^t\,dt.$ Use parts for each integral (I'll do them one at a time):

$\int_0^1 te^t\,dt = te^t - \int e^t\,dt = te^t - e^t\Big]_0^1 = (\cancel{e} - \cancel{e}) - (0-1) = 1.$

$u = t \quad v = e^t$
$du = dt \quad dv = e^t\,dt$

$\int_0^1 t^2e^t\,dt = t^2e^t\Big]_0^1 - 2\underbrace{\int_0^1 te^t\,dt}_{\substack{\text{just did}\\\text{this integral}\\(=1)}} = t^2e^t\Big]_0^1 - 2\cdot 1 = (e-0) - 2 = e - 2.$

$\uparrow$ from above

$u = t^2 \quad v = e^t$
$du = 2t\,dt \quad dv = e^t\,dt$

So, $\int_0^1 te^t\,dt - \int_0^1 t^2e^t\,dt = 1 - (e-2) = 1 - e + 2 = \boxed{3-e}$.

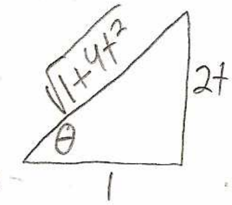
37. Set up an integral that represents the length of the curve. Then use a calculator to find the length correct to four decimal places.

$x = t - t^2,\ y = \frac{4}{3}t^{\frac{3}{2}},\ 1 \le t \le 2 \qquad \frac{dx}{dt} = 1 - 2t \quad \frac{dy}{dt} = \frac{3}{2}\cdot\frac{4}{3}t^{\frac{1}{2}} = 2\sqrt{t}$

$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^2 \sqrt{(1-2t)^2 + (2\sqrt{t})^2}\,dt = \int_1^2 \sqrt{1 - \cancel{4t} + 4t^2 + \cancel{4t}}\,dt$$

$$= \boxed{\int_1^2 \sqrt{1+4t^2}\,dt \approx 3.1678}$$

Can use trig sub. to do the integral:



$$\int_1^2 \sqrt{1+4t^2}\,dt = \frac{1}{2}\int \sec^3\theta\,d\theta = \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]$$

$2t = \tan\theta$

$2dt = \sec^2\theta\,d\theta \rightarrow dt = \frac{1}{2}\sec^2\theta\,d\theta$

$\sqrt{1+4t^2} = \sec\theta$

$$= \frac{1}{4}\left[\sqrt{1+4t^2}\cdot 2t + \ln\left|\sqrt{1+4t^2} + 2t\right|\right]_1^2$$

$$= \frac{1}{4}\left[\left(4\sqrt{17} + \ln\left(\sqrt{17}+4\right)\right) - \left(2\sqrt{5} + \ln\left(\sqrt{5}+2\right)\right)\right]$$

$$= \frac{1}{4}\left[4\sqrt{17} + \ln\left(\sqrt{17}+4\right) - 2\sqrt{5} - \ln\left(\sqrt{5}+2\right)\right] \approx 3.1678$$

39. Set up an integral that represents the length of the curve. Then use a calculator to find the length correct to 4 decimal places.

$x = t + \cos t,\ y = t - \sin t,\ 0 \le t \le 2\pi$

$\frac{dx}{dt} = 1 - \sin t \quad \frac{dy}{dt} = 1 - \cos t$

$\left(\frac{dx}{dt}\right)^2 = 1 - 2\sin t + \sin^2 t \quad \left(\frac{dy}{dt}\right)^2 = 1 - 2\cos t + \cos^2 t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{1 - 2\sin t + \underbrace{\sin^2 t + 1 - 2\cos t + \cos^2 t}_{=1}}\,dt$$

$$= \boxed{\int_0^{2\pi} \sqrt{3 - 2\sin t - 2\cos t}\,dt \approx 10.0367}$$

41. Find the exact length of the curve $x = 1 + 3t^2,\ y = 4 + 2t^3,\ 0 \le t \le 1$

$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{36t^2 + 36t^4}\,dt$

$\frac{dx}{dt} = 6t,\ \left(\frac{dx}{dt}\right)^2 = 36t^2.\ \frac{dy}{dt} = 6t^2,\ \left(\frac{dy}{dt}\right)^2 = 36t^4.$

$$= \int_0^1 \sqrt{36t^2(1+t^2)}\,dt = \frac{1}{2}\int_0^1 6t\sqrt{1+t^2}\,dt\cdot 2 = 3\int_1^2 u^{\frac{1}{2}}\,du = \cancel{3}\cdot\frac{2}{\cancel{3}}u^{\frac{3}{2}}\Big]_1^2$$

$u = 1 + t^2,\ du = 2t\,dt$

t	u
0	1
1	2

$$= 2\cdot\left[2^{\frac{3}{2}} - 1\right]$$

$$= \boxed{2\left[2\sqrt{2} - 1\right]}$$

57. Set up an integral that represents the area of the surface obtained by rotating the curve about the x-axis. Then use a calculator to find the S.A. correct to 4 decimal places.

$x = 1 + te^t,\ y = (t^2+1)e^t,\ 0 \le t \le 1$

$\frac{dx}{dt} = e^t + te^t,\ \left(\frac{dx}{dt}\right)^2 = \left[e^t(1+t)\right]^2 = e^{2t}(1+t)^2 = e^{2t}(1+2t+t^2)$

$\frac{dy}{dt} = 2te^t + (t^2+1)e^t,\ \left(\frac{dy}{dt}\right)^2 = \left[e^t(2t+t^2+1)\right]^2 = e^{2t}\left[(t+1)^2\right]^2 = e^{2t}(t+1)^4$

$$S.A. = \int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_0^1 (t^2+1)e^t\sqrt{e^{2t}(t+1)^2 + e^{2t}(t+1)^4}\,dt$$

$$= 2\pi\int_0^1 (t^2+1)e^t\sqrt{e^{2t}(t+1)^2\left[1+(t+1)^2\right]}\,dt = 2\pi\int_0^1 (t^2+1)e^t\cdot e^t(t+1)\sqrt{1+(t+1)^2}\,dt$$

$$= 2\pi\int_0^1 (t^2+1)e^{2t}(t+1)\sqrt{1+t^2+2t+1}\,dt = 2\pi\int_0^1 (t^2+1)e^{2t}(t+1)\sqrt{t^2+2t+2}\,dt$$

$$\boxed{\approx 103.5999}$$

☆ Key formulas for parametric curves:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Horizontal tangent line: $\frac{dy}{dt} = 0$

Vertical tangent line: $\frac{dx}{dt} = 0$

Equation of tangent line:

$y - y_1 = \underset{\frac{dy}{dx}}{m}(x - x_1)$

Area $= \int_{x_1}^{x_2} y\,dx = \int_{t_1}^{t_2}$ (whatever you get when you substitute for y and dx)

Length of curve $= \int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Surface Area when curve is rotated about the x-axis $= \int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$