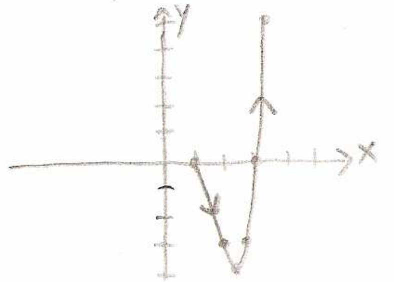
10.1 homework parametric equations

1. $x = 1 + \sqrt{t}$, $y = t^2 - 4t$, $0 \le t \le 5$

t	x	y
0	1	0
1	2	-3
2	2.4	-4
3	2.7	-3
4	3	0
5	3.2	5

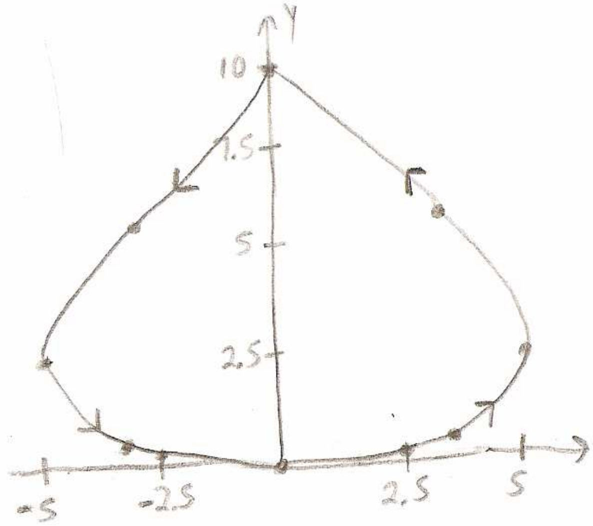


3. $x = 5\sin t$, $y = t^2$ $\quad -\pi \le t \le \pi$

t	x	y
$-\pi$	0	$\pi^2 = 9.9$
$-\frac{\pi}{2}$	-5	$\frac{\pi^2}{4} \approx 2.5$
0	0	0
$\frac{\pi}{2}$	5	2.5
π	0	9.9

(I got a few more points over here)

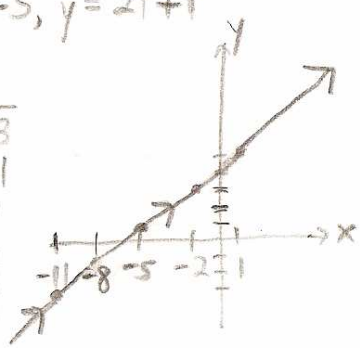
t	x	y
$-\frac{\pi}{4}$	$5\left(-\frac{\sqrt{2}}{2}\right) = -3.5$	$\frac{\pi^2}{16} = .6$
$\frac{\pi}{4}$	$5\left(\frac{\sqrt{2}}{2}\right) = 3.5$	.6
$-\frac{\pi}{6}$	-2.5	$\frac{\pi^2}{36} = .3$
$\frac{\pi}{6}$	2.5	.3
$-\frac{3\pi}{4}$	$5\left(-\frac{\sqrt{2}}{2}\right) = -3.5$	$\frac{9\pi^2}{16} = 5.6$
$\frac{3\pi}{4}$	$5\left(\frac{\sqrt{2}}{2}\right) = 3.5$	5.6



5. $x = 3t - 5$, $y = 2t + 1$

a)

t	x	y
-2	-11	-3
-1	-8	-1
0	-5	1
1	-2	3
2	1	5



b) $\frac{x+5}{3} = t$, so $y = 2 \cdot \frac{(x+5)}{3} + 1$

$y = \frac{2}{3}x + \frac{10}{3} + \frac{3}{3}$

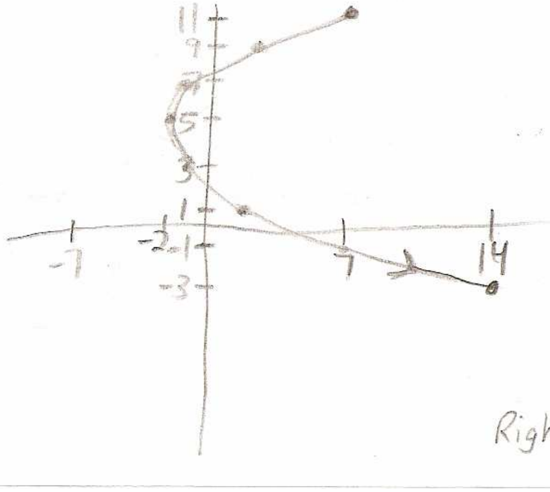
$y = \frac{2}{3}x + \frac{13}{3}$.

Line with slope $\frac{2}{3}$, y-int. $(0, 4\frac{1}{3})$.

7. $x = t^2 - 2$, $y = 5 - 2t$, $-3 \le t \le 4$

a)

t	x	y
-3	7	11
-2	2	9
-1	-1	7
0	-2	5
1	-1	3
2	2	1
3	7	-1
4	14	-3



b) $y = 5 - 2t \Rightarrow 2t = 5 - y$

$t = \frac{5-y}{2}$

$x = t^2 - 2 \Rightarrow x = \left(\frac{5-y}{2}\right)^2 - 2$

$\Rightarrow x = \frac{1}{4}(5-y)^2 - 2$

OR $x = \frac{1}{4}(y-5)^2 - 2$

OR $x + 2 = \frac{1}{4}(y-5)^2$

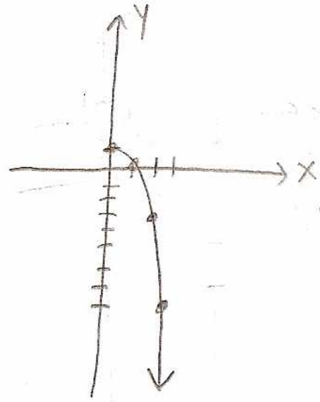
$(-3 \le y \le 11)$

Right-opening parabola piece, vertex = $(-2, 5)$

9. $x=\sqrt{t}$, $y=1-t$

a)

t	x	y
0	0	1
1	1	0
4	2	$1-4=-3$
9	3	$1-9=-8$
16	4	$1-16=-15$



b) $x=\sqrt{t} \Rightarrow x^2=t$

$y=1-t \Rightarrow y=1-x^2$

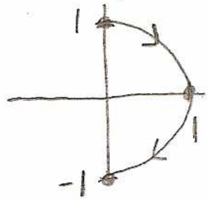
$y=-x^2+1 \quad (x\geq 0)$

Right half of down-opening parabola shifted up 1.

11. $x=\sin\theta$, $y=\cos\theta$, $0\leq\theta\leq\pi$

a) $\sin^2\theta+\cos^2\theta=1$, so $x^2+y^2=1$.

b)



θ	x	y
0	0	1
$\frac{\pi}{2}$	1	0
π	0	-1

Right half of circle.

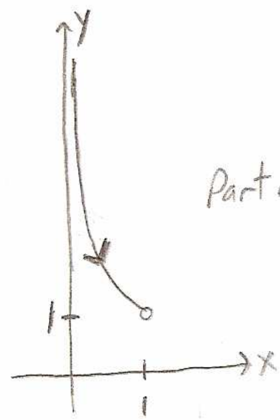
$0\leq x\leq 1$

$-1\leq y\leq 1$

13. $x=\sin t$, $y=\csc t$, $0<t<\frac{\pi}{2}$

a) $y=\frac{1}{\sin t} \Rightarrow y=\frac{1}{x}$ $\quad 0<x<1$

b)



Part of the graph of $y=\frac{1}{x}$

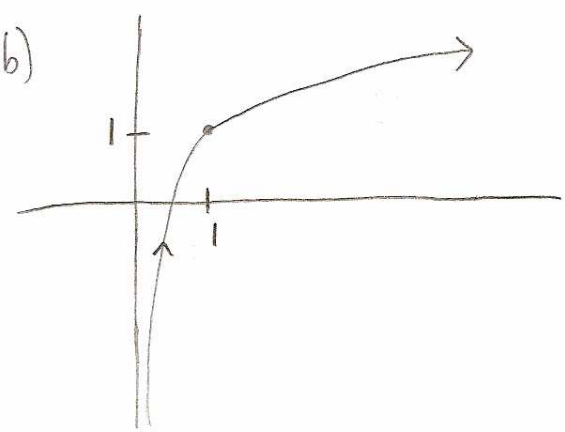
t	x	y
$\frac{\pi}{6}$	$\frac{1}{2}$	2
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}=.7$	$\sqrt{2}=1.4$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}=.87$	$\frac{2}{\sqrt{3}}=1.2$

15. $x=e^{2t}$, $y=t+1$

a) $\ln x=2t \Rightarrow t=\frac{1}{2}\ln x \Rightarrow y=\frac{1}{2}\ln x+1$

t	x	y
0	1	1
1	e^2	2

b)



17. $x=\sinh t$, $y=\cosh t$

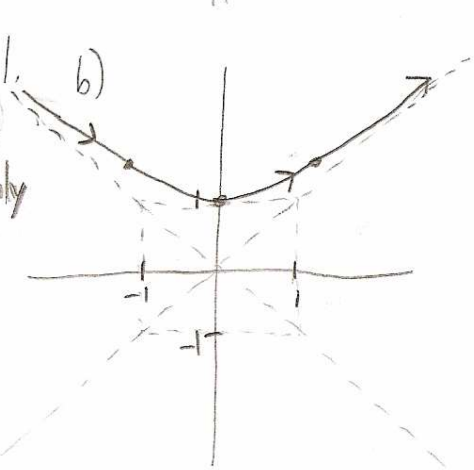
a) From section 3.11, $\cosh^2 t-\sinh^2 t=1$, so $y^2-x^2=1$.

hyperbola; top branch only

$x=\frac{e^x-e^{-x}}{2}$, $y=\frac{e^x+e^{-x}}{2}$

t	x	y
-1	$\frac{e^{-1}-e^{1}}{2}\approx-1.2$	$\frac{e^{-1}+e^{1}}{2}\approx 1.5$
0	$\frac{e^0-e^0}{2}=0$	$\frac{e^0+e^0}{2}=\frac{1+1}{2}=1$
1	$\frac{e^1-e^{-1}}{2}\approx 1.2$	$\frac{e^1+e^{-1}}{2}\approx 1.5$

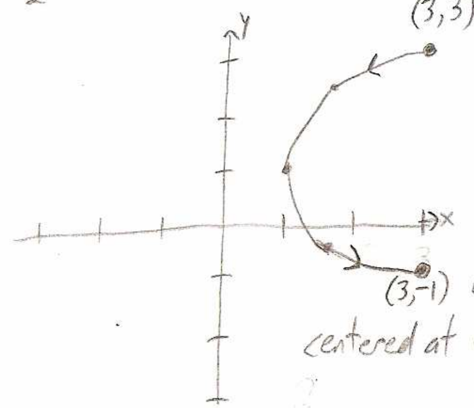
b)



10.1 homework

19. $x = 3+2\cos t,\ y = 1+2\sin t,\ \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

t	x	y
$\frac{\pi}{2}$	$3+2\cdot 0 = 3$	$1+2\cdot 1 = 3$
$\frac{3\pi}{4}$	$3+\cancel{2}\cdot\left(-\frac{\sqrt{2}}{\cancel{2}}\right) \approx 1.6$	$1+\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}} \approx 2.4$
π	$3+2\cdot(-1) = 1$	$1+2\cdot 0 = 1$
$\frac{5\pi}{4}$	$3+\cancel{2}\left(-\frac{\sqrt{2}}{\cancel{2}}\right) \approx 1.6$	$1+\cancel{2}\left(-\frac{\sqrt{2}}{\cancel{2}}\right) = -.4$
$\frac{3\pi}{2}$	$3+2\cdot 0 = 3$	$1+2\cdot(-1) = -1$



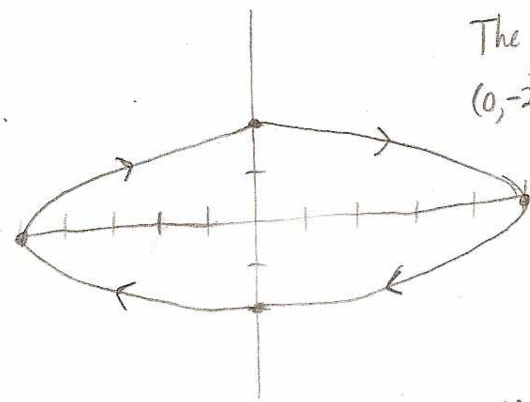
The particle moves counter-clockwise around the left half of a circle centered at (3,1) with radius 2.

21. $x = 5\sin t,\ y = 2\cos t,\ -\pi \le t \le 5\pi$

$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \Rightarrow \frac{x^2}{25} + \frac{y^2}{4} = 1$ ellipse

t	x	y
$-\pi$	$5\cdot 0 = 0$	$2\cdot(-1) = -2$
$-\frac{\pi}{2}$	$5(-1) = -5$	$2\cdot 0 = 0$
0	$5\cdot 0 = 0$	$2\cdot 1 = 2$
$\frac{\pi}{2}$	$5\cdot 1 = 5$	$2\cdot 0 = 0$
π	$5\cdot 0 = 0$	$2\cdot(-1) = -2$

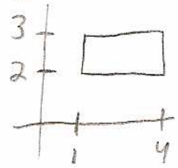
repeats every 2π



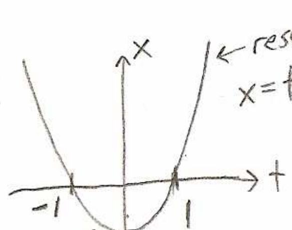
The particle begins at $(0,-2)$ and makes 3 trips clockwise around the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$.

23. Suppose a curve is given by the parametric equations $x = f(t)$ and $y = g(t)$, where the range of f is $[1,4]$ and the range of g is $[2,3]$. What can you say about the curve?

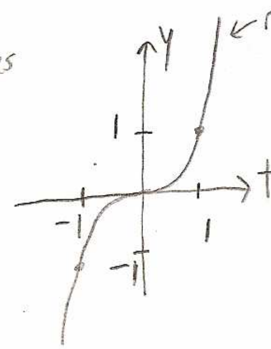
Answer: The "range" of f is the set of x values, so $1 \le x \le 4$. The "range" of g is the y values, so $2 \le y \le 3$. The curve must be contained within the rectangular region shown:



25. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t),\ y = g(t)$.



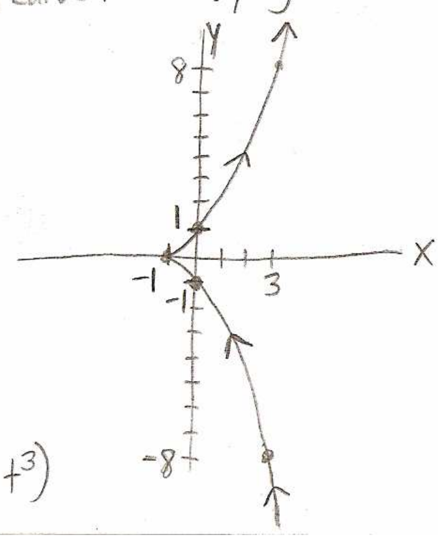
← resembles $x = t^2 - 1$



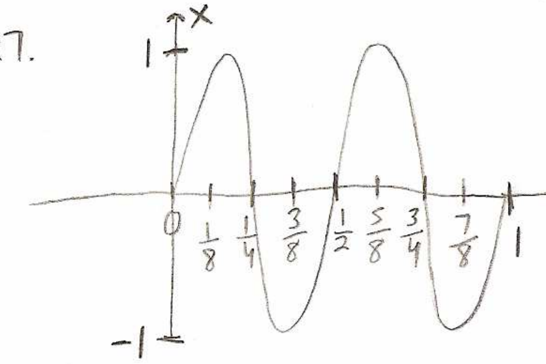
← resembles $y = t^3$

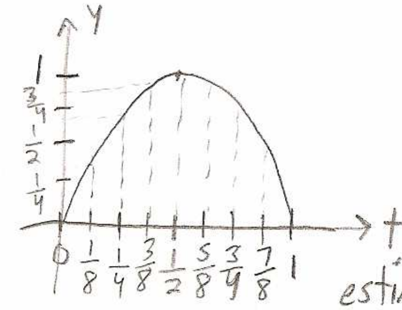
t	x	y
-3	8	-27
-2	3	-8
-1	0	-1
0	-1	0
1	0	1
2	3	8
3	8	27

estimate the x and y values from the graphs and the equations that appear to describe them ($x = t^2 - 1$ and $y = t^3$)



27.

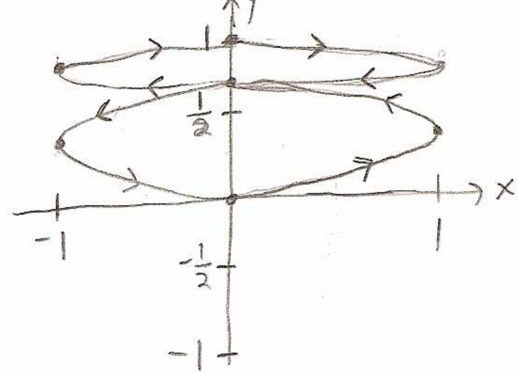




estimating:

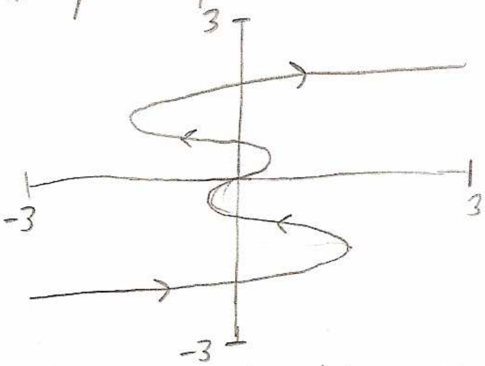
t	x	y
0	0	0
$\frac{1}{8}$	1	.3
$\frac{1}{4}$	0	.7
$\frac{3}{8}$	-1	.9
$\frac{1}{2}$	0	1
$\frac{5}{8}$	1	.9
$\frac{3}{4}$	0	.7
$\frac{7}{8}$	-1	.3
1	0	0

Use the graphs of $x=f(t)$ and $y=g(t)$ to sketch the parametric curve $x=f(t)$, $y=g(t)$.



29. Graph $x=y-3y^3+y^5$ on the calculator.

To do it, use $x=t-3t^3+t^5$ and $y=t$ in parametric mode, with $t_{min}=-3$ and $t_{max}=3$:



31. Show that the parametric equations $x=x_1+(x_2-x_1)t$, $y=y_1+(y_2-y_1)t$, where $0\le t\le 1$, describe the line segment that joins the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$.

t	x	y
0	x_1	y_1
1	$\cancel{x_1}+x_2-\cancel{x_1}=x_2$	$\cancel{y_1}+y_2-\cancel{y_1}=y_2$

← This shows that the line segment begins at (x_1,y_1) and ends at (x_2,y_2).

b. Find parametric equations to represent the line segment from $(\underset{x_1}{-2},\underset{y_1}{7})$ to $(\underset{x_2}{3},\underset{y_2}{-1})$.

Use $x=x_1+(x_2-x_1)t$ and $y=y_1+(y_2-y_1)t$:

$x=-2+(3-(-2))t \qquad y=7+(-1-7)t$

$x=-2+5t$, $y=7-8t$ (for $0\le t\le 1$)

45. Suppose that the position of one particle at time t is given by

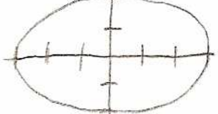
$x_1 = 3\sin t \quad y_1 = 2\cos t \quad 0 \le t \le 2\pi$

and the position of a second particle is given by

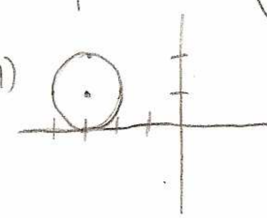
$x_2 = -3 + \cos t \quad y_2 = 1 + \sin t \quad 0 \le t \le 2\pi$

a) Graph the paths of both particles. How many points of intersection are there?

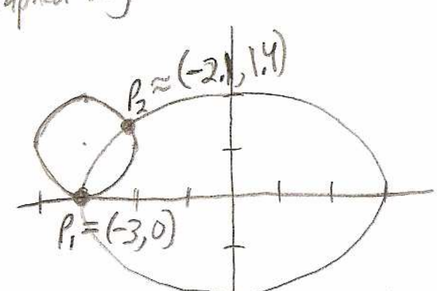
1st particle: $\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$ ellipse



2nd particle: $(x_2+3)^2 + (y_2-1)^2 = 1$ circle
center $= (-3,1)$
radius $= 1$



$\Rightarrow$ graphed together:



there are 2 intersection points.

b) Is either of the two intersection points a "collision point"? If it is, we must have:

$x_1 = x_2$ and $y_1 = y_2$ for the same t value.

$3\sin t = -3 + \cos t$

Notice that for $t = \frac{3\pi}{2}$: $3\sin\frac{3\pi}{2} = -3$
and $-3 + \cos\frac{3\pi}{2} = -3$,

$2\cos t = 1 + \sin t$

for $t = \frac{3\pi}{2}$: $2\cos\frac{3\pi}{2} = 0$
and $1 + \sin\frac{3\pi}{2} = 0$

$\Rightarrow$ The point $(-3,0)$ is a collision point, since the particles are both there when $t = \frac{3\pi}{2}$.

Here is a way to get $\frac{3\pi}{2}$ algebraically

The system of equations $\begin{cases} 3\sin t = -3 + \cos t & \text{(Eq.1)} \\ 2\cos t = 1 + \sin t & \text{(Eq.2)} \end{cases}$ can be solved by substitution, as follows...

solve Eq.2 for $\sin t$: $\sin t = 2\cos t - 1$. Substitute into Eq.1: $3(2\cos t - 1) = -3 + \cos t$

$6\cos t - 3 = -3 + \cos t$

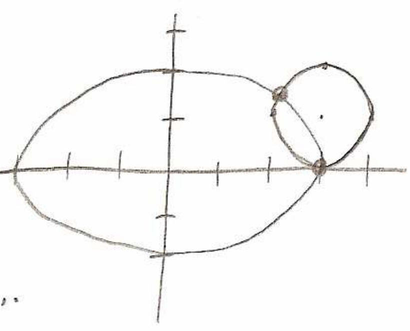
$5\cos t = 0 \Rightarrow \cos t = 0$.

$\cos t = 0$ for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. $t = \frac{3\pi}{2}$ satisfies both equations, while $t = \frac{\pi}{2}$ does not.

c) Describe what happens if the path of the second particle is given by $x_2 = 3 + \cos t$, $y = 1 + \sin t$, $0 \le t \le 2\pi$

$(x-3)^2 + (y-1)^2 = 1$ circle, center $(3,1)$, radius 1.

The intersection pts. in this case are $(3,0)$ and $\approx (2.1, 1.4)$.



To see if there are any collision points (next page)...

we would need $x_1 = x_2$ and $y_1 = y_2$ for the same t value:

$$3\sin t = 3 + \cos t \qquad 2\cos t = 1 + \sin t \Rightarrow \boxed{2\cos t - 1} = \sin t$$

$$3(2\cos t - 1) = 3 + \cos t$$

$$6\cos t - 3 = 3 + \cos t$$

$$5\cos t = 6$$

$\cos t = \frac{6}{5} > 1$, so this equation has no solution $\Rightarrow$ there are no collision points.